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LIGHT AND SOUND

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LIGHT AND SOUND

A TEXT BOOK FOR COLLEGES AND TECHNICAL SCHOOLS

BY
WILLIAM S.uddards FRANKLIN
PROFESSOR OF PHYSICS IN THE MASSACHUSETTS
INSTITUTE OF TECHNOLOGY

AND

BARRY MacNUTT
PROFESSOR OF PHYSICS IN LEHIGH UNIVERSITY

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LIGHT AND SOUND

CHAPTER I

LIGHT AND SOUND DEFINED.* VELOCITY OF TRANSMISSION

1. Sensory nerves. The sensory nerves of the human body lead from regions near the surface of the body to the central organs of the nervous system. The outer ends of these nerves are exposed in such a way as to be excited or set into commotion by physical disturbances in the region surrounding the body, this commotion is transmitted to the central organs producing commotion there, and we experience what we call a *sensation.* The physical disturbance which excites the nerves is called a *stimulus.*

2. Proper stimuli. *End organs. Localization.* Each set of sensory nerves, such as the nerves of sight or the nerves of smell, is especially sensitive to a certain kind of disturbance, and the disturbance to which a given set of sensory nerves is especially sensitive is called the *proper stimulus* of that set of nerves.

A set of sensory nerves is rendered especially sensitive to its proper stimulus in two distinct ways, namely, (*a*) by being provided with *specialized end organs*, and (*b*) by being *located* so as to be exposed to the proper stimulus but protected to a great extent from other stimuli. Thus the nerves of sight terminate in minute organs, the so-called rods and cones, which are situated in the retina of the eye. These organs are sensitive to the disturbances which can reach them through the transparent humors of the eye, and they are to a great extent protected from all other physical disturbances.

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* It is convenient in this chapter to treat both light and sound.

When a given set of sensory nerves is excited, *the sensation which corresponds to the set of nerves is always produced** no matter what the character of the stimulus may be. Thus, a sensation of light is always produced when the nerves of sight are excited, be the excitation caused by a severe mechanical shock, by an electric current passing through the eye, by abnormal conditions of blood circulation through the retina, or by the proper stimulus (the light which passes through the transparent humors of the eye. See next article).

3. Light, the sensation; light, the proper stimulus. The sensation which is experienced when the nerves of sight are excited is called *light*. That physical disturbance which constitutes the proper stimulus of the nerves of sight is also called *light*. The most familiar property of the physical disturbance which is called light is that it can pass through many substances, such as glass and water; that is to say, we can "see through" such substances, and they are said to be *transparent*.

The study of light as a sensation belongs to the subject of psychology; the study of the physical disturbance which constitutes the proper stimulus of the nerves of sight belongs to the subject of physics. This treatise is devoted to the study of light, the proper stimulus.

4. Nerves of hearing. *Sound, the sensation; sound, the proper stimulus.* The nerves of hearing terminate in end organs which float in a watery fluid which is contained in a bone-walled cavity called the *inner ear*. These end organs are to a great extent protected by the massive bones of the head from all disturbances except vibratory movements of the air which reach them as follows: Figure 1 shows the essential features of the ear.* The cavity C which is filled with watery fluid constitutes the *inner ear*, and it is provided with two "windows," the *oval window* O and the *round window* R which are closed by very thin membranes. The *middle ear* M is an air-filled cavity which communicates with the mouth cavity through the *Eustachian tube* E, and three small bones in the form of a chain bridge across from the *ear drum* D to the membrane which covers the oval window. The vibratory motion of the outside air causes the ear drum D to vibrate. These vibrations are transmitted to the oval window by the chain of small bones, the vibration of the membrane of the oval window causes the fluid in the inner ear to surge back and forth through the complicated channels of the inner ear between the oval window and

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* This is the so-called "principle of the specific energy of the nerves." It is now known to be inconsistent with some of the known facts of sensation.

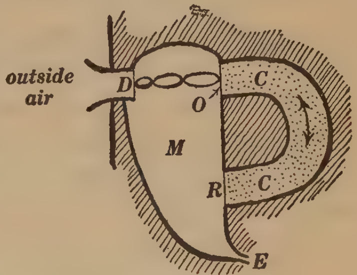

Fig. 1.

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* This figure is not intended to show the actual structure but to illustrate the principles of action of the ear. No attempt has been made in this figure to show the cavity of the inner ear (the *cochlea*) in which the *basilar membrane* is situated, which contains the sense organs of musical pitch. The principles of action of the basilar membrane are explained in Chapter XV. A very full discussion of the structure and action of the ear may be found in Helmholtz's *Tonempfindungen*, pages 189–226 of the English translation by Alexander J. Ellis (Longmans & Co.).

the round window, as indicated by the double-headed arrow in Fig. 1, and the end organs of the nerves of hearing are excited by this surging fluid.

The sensation which is experienced when the nerves of hearing are excited is called *sound*. That physical disturbance which constitutes the proper stimulus of the nerves of hearing is also called *sound*. The study of sound, the sensation, belongs to the science of psychology. This treatise is devoted to the study of sound, the proper stimulus.

5. Transmission of light and sound. We have come by experience to associate distant objects with our sensations of light and sound—that is, we *see* and *hear* distant objects—and the attempt to explain the evident connection between a distant object and the sensations of light and sound which it produces has led philosophers to assume the existence of a physical agent which we call *light* and which is transmitted to our eyes from a distant object; and to assume the existence of a physical agent which we call *sound* and which is transmitted to our ears from a distant object.

6. The corpuscular theory of light.* The phenomena of shadows and the obstruction of vision of a distant object by intervening objects show that light travels sensibly in straight lines. In accordance with this fact it was the accepted theory, until long after the time of Sir Isaac Newton, that light consisted of particles or corpuscles which were thrown off from luminous bodies at great velocity, traveling in straight lines until reflected or stopped by objects upon which they impinged. This was called the *corpuscular theory of light*.

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* A very interesting discussion of the corpuscular theory of light is to be found in Preston's *Theory of Light*, pages 15–21. Sir Isaac Newton was the greatest exponent of the corpuscular theory.

7. The wave theory of light.* Most of the phenomena of light have been rationalized (made understandable) by considering light to be a wave-like disturbance which travels out in all directions from a luminous body, and this wave-like disturbance is supposed to travel in a special transmitting medium, the *ether*, which fills all space. No other theory of light has ever been proposed which compares with the wave theory in usefulness, and therefore the wave theory of light is used throughout this text.

Recent developments of the principle of relativity of motion† seem, however, to show the non-existence of the ether as a wave transmitting medium, and the new quantum hypothesis,‡ which has been used with very remarkable success in rationalizing the phenomena of radiation, seems to show that light energy cannot be spread out thin, as it would have to be in a wave, but must be concentrated in small pieces or quanta !

8. The wave theory of sound. The physical nature of sound is a matter of fact, not a matter of hypothesis.

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* See "The wave theory of light," a collection of original memoirs by Huygens, Young, and Fresnel, translated and arranged by Henry Crew. The Scientific Memoirs Series, edited by J. S. Ames.

A very interesting semi-popular book is *Light Waves and their Uses*, by A. A. Michelson, Chicago University Press, 1903.

† See *Relativity and the Electron Theory*, by E. Cunningham, Longmans, Green, London, 1921.

‡ A good discussion of the quantum theory is given by J. H. Jeans in a *Report on Radiation and the Quantum Theory*, published by the London Physical Society, 1914.

A brief discussion of the quantum theory is given by Norman Campbell in a Supplement to his *Modern Electrical Theory*, published as Chapter XV, by the Cambridge University Press, 1921.

A very full discussion of the application of the quantum theory to the origin of spectra is given in a very recent book entitled *The Origin of Spectra*, by Paul D. Foote and F. L. Mohler, American Chemical Society Monograph No. 8, September, 1922. See page 133.

Ordinarily sound consists of wave disturbances in the air; a sounding body vibrates, these vibrations set the adjacent air in motion, and air-waves spread out in all directions from the sounding body very much as water waves spread out from a point where a stone is thrown into a pond.

Sound waves are also transmitted through water and through solid bodies.

9. The velocity of sound.* It is a familiar fact that sound requires a perceptible time to reach the ear from a sounding body. The first attempt to measure accurately the velocity of sound was made by a committee of members of the French Academy of Sciences in 1738. The observers were placed at the Paris Observatory and at three distant stations visible from the observatory. Every ten minutes a cannon was fired at one of the stations, and the observers at the other stations noted the time intervals which elapsed between the flash and the sound of the cannon. The light flash was transmitted almost instantaneously, so that the observed time intervals were taken to be the time intervals required for the sound to travel over the measured distances between the stations.

In 1822 this experiment was repeated at Paris in a slightly modified form. Two stations were selected, cannon were fired at these stations alternately, and the time intervals between flash and sound were observed as before. By firing at the two stations alternately the influence of the wind was eliminated. The distance between the stations was 18,622 meters (about 10 miles). The observed time interval between flash and sound at one station was 54·84 seconds and at the other station it was 54·43 seconds. These data gave 340·9 meters per second as the velocity of sound; allowing for the temperature of the air at the time the observations were taken, this corresponds to 331·2 meters per second at 0° C. More recent determinations of the velocity of sound give 331 meters per second for the velocity in dry air at 0° C. The velocity of sound in air depends upon the temperature and it also depends upon humidity.

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* The velocity of sound varies quite perceptibly with the loudness. See *Encyclopædia Britannica*, 9th edition, article *Acoustics*, sections 20 to 27. The velocity of sound varies greatly with the temperature of the air. This matter is discussed in the *Britannica* article. See also Poynting and Thomson's *Text-Book of Physics*, Volume on *Sound*, pages 16–31.

The following table gives the velocity of sound in various substances in terms of the velocity of sound in air:

TABLE

Velocities of Sound in Different Media

Air	Unity
Iron	15·1
Glass	15·3
Water	4·3

10. Velocity of light.* A Danish astronomer, Roemer (1675), was the first to show that light has a finite velocity. He found that the observed time of revolution of the satellites of Jupiter varies with the position of the earth in its orbit. When the earth is moving towards Jupiter the observed time of revolution of a satellite is less than the true time of revolution, and when the earth is moving away from Jupiter the observed time is greater than the true time; the true time of revolution being the mean of all the observed times during one revolution of the earth in its orbit. The cause of the variation of the observed time of revolution of Jupiter's satellites is as follows:

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* For full account of the researches upon the velocity of light, see Preston's *Theory of Light*, Chapter XIX.

Suppose a light signal is flashed at equal intervals of time. A stationary observer sees these flashes separated by the true time interval between them. If, during the interval between two flashes, the observer moves towards or away from the source of the flashes, the interval between observed flashes will be less or more than the true interval and the amount of this difference will be equal to the time required for light to pass over the distance that the observer has moved.

A laboratory method for measuring the velocity of light was devised by Fizeau in 1849. This method was employed under more favorable conditions by Cornu in 1874. Another method was devised by Foucault in 1850. This method has been used by various observers, notably by Michelson in 1880. Newcomb in 1882 carried out Foucault's method. He found, from an extended series of observations, a velocity of 299,778,000 meters per second in air (299,860,000 meters per second in vacuum). This is perhaps the most reliable determination that has been made.

LEADING QUESTIONS

L. Q. 1. When the points of a compass are less than about 0·18 cm. apart the two points are perceived as a single point when applied to the end of the finger, and when the points are less than 1·2 cm. apart they are perceived as a single point when applied to the back of the hand. What can be said as to the size of the spots on the tip of the finger and on the back of the hand which are sensitized by single nerve fibers?

L. Q. 2. Two points of light at a distance of 1000 feet are seen as a single point if they are nearer together than 3 or 4 inches. In this case the images of the two points on the retina are about 0·00025 inch apart. What can be said as to the size of the spot on the retina which is sensitized by a single nerve fiber?

L. Q. 3. What about the truth of the following statements? (*a*) The most terrific explosion would produce no sound if no one were present to hear it. (*b*) The most brilliant day would be dark as night if no one were present to see the light.

PROBLEMS

1. Ten seconds elapse between a flash of lightning and the first peal of thunder, and the long-drawn-out sound of thunder lasts six seconds. How far away from the listener is the nearest part of the lightning, and how far away from the listener is the most distant part of the lightning?

Note. The long-drawn-out rumble of thunder may conceivably be due to reflections of sound (echoes), but in a non-mountainous country it is due almost wholly to the fact that the sounds from the various parts of a long lightning flash reach the listener's ear in succession as a long-drawn-out rumble.

2. Two listening stations A and B are located back of the front line in a battle and the sound of an enemy gun is heard at A 0·25 second after it is heard at B. What is the value of the difference $AG - BG$, where AG is the distance from A to the gun and BG is the distance from B to the gun?

Note. If the distance $AG - BG$ is known, the gun must lie on a known hyperbola whose foci are at A and B. If a third listening station C is used the gun can be located on a known hyperbola whose foci are A and C, and thus the gun must be at the place where the two hyperbolas intersect. This problem illustrates the principles used in "sound ranging."

3. A man on a car makes four strokes of a hammer per second while the car is passing a stationary observer or listener. Find the frequency of the strokes *as heard by the listener*, (*a*) while the car is approaching the listener, and (*b*) while the car is moving away from the listener; the velocity of the car being 90 feet per second. Ans. (*a*) 4·33 strokes per second; (*b*) 3·67 strokes per second.

CHAPTER II

SOME USEFUL IDEAS OF WAVE MOTION*

11. Wave-pulses and wave-trains. When a stone is pitched into a pond a wave emanates from the place where the stone strikes. When a long stretched wire is struck sharply with a hammer, a single wave (a bend in the wire) travels along the wire in both directions from the point where the wire is struck. When a long steel rod is struck on the end with a hammer, a single wave (an endwise compression of the rod) travels along the rod. When an explosion takes place in the air, the firing of a gun for example, a single wave (a compression of the air) travels outwards from the explosion. Such isolated waves are called *wave-pulses.*

When a disturbance at a point in a medium is repeated in equal intervals of time the disturbance is said to be *periodic.* Such a disturbance sends out a succession of similar waves, constituting what is called a *wave-train.*

The distance between similar parts of two successive waves of a train is called the *wave-length* of the train. If V is the velocity of travel of the waves and τ the time which elapses between repetitions of the periodic disturbance which produces the wave-train, then $V\tau$ is the distance traveled by one wave when the next succeeding wave is starting out. Therefore

$$\lambda = V\tau \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

where λ is the wave-length of the wave-train.

---

* The discussion of standing waves and the discussion of reflection with and without phase reversal are given in Chapter XIII.

Transverse waves and longitudinal waves. Holding one end of a long stretched rope in the hand, move the end of the rope to and fro sidewise and a train of waves will be seen to travel along the rope. Every particle of the rope moves to and fro sidewise (at right angles to the rope) as the wave-train travels along, and because of this transverse motion the waves on the rope are called *transverse waves.*

Imagine a closely fitting piston to be moved rapidly in and out near one end of a long tube filled with air. This motion of the piston produces a train of waves which travels along in the air in the tube, every particle of air in the tube moves to and fro in a direction parallel to the tube as the wave-train travels along, and because of this longitudinal motion (motion to and fro in the direction of travel of the waves) the waves in the air in the tube are called *longitudinal waves.*

Light waves are known to be transverse waves because transverse waves, only, can be polarized (see Chapter IX). Sound waves in air are longitudinal waves.

12. Doppler's principle. Wave-length of sound or light emitted by a moving source. A tuning fork makes n complete vibrations per second (or one vibration in $\tau = 1/n$th of a second), the sound waves from the fork travel at velocity V through the surrounding still air, and the wave-length of the wave-train which is emitted by the fork is $\lambda = V\tau = V/n$.

Let us suppose that the tuning-fork is moving to the right at velocity v, as indicated in Fig. 2, and let us consider the effect of this motion of the fork on the wave-length of the wave-train which travels to the right from the fork at velocity V. Let A be the position of the fork at a given instant, and let B be the position of the fork t

seconds later. The wave which started out from the fork when the fork was at A will travel the distance Vt during the t seconds and be at WW when the fork is at B, and the nt waves which leave the fork during the t seconds will all be crowded between B and WW. Therefore if we divide the distance BW, which is $Vt - vt$, by nt we will get the desired wave-length λ', and if we wish to find the frequency n' with which the waves from the moving fork strike a stationary observer's ear at W we consider the fact that λ' is the distance the waves travel in $1/n'$th of a second at velocity V, so that $\lambda' = V \times 1/n'$ or $n' = V/\lambda'$. A precisely similar argument will give the values of λ' and n' if v is towards the left in Fig. 2.

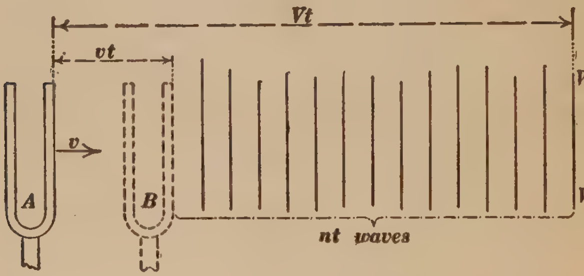

Fig. 2.

Remark. Doppler's principle applies to trains of light waves as well as to trains of sound waves.

13. The wave front. One of the most important ideas in connection with the wave theory of light is the idea of the *wave front.* Every one is familiar with the fact that waves on the surface of a pond always resolve themselves into clearly defined ridges at a distance from the disturbance, however complicated the disturbance may be. Thus, when a handful of pebbles is thrown into a pond the wave motion in the immediate neighborhood of the disturbance is excessively complicated, but the waves become a clearly defined series of ridges at a considerable distance from the disturbance.

Consider a region *AB*, Fig. 3, on the surface of a pond, *C* being a point at which a handful of pebbles is dropped into a pond. The above-mentioned fact that the waves from *C* resolve themselves into clearly defined ridges at a distance from *C* means that *all points of the water surface which lie in a certain line AB* **rise and fall together,** *or, in other words, the line AB on the water surface moves up and down as a whole as the waves pass by.* Such a line is called a *wave front.*

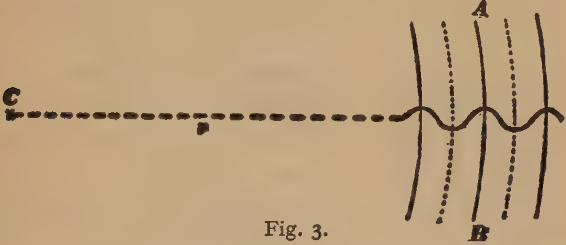

Fig. 3.

Sound waves in the air and light waves in the ether always resolve themselves at a great distance from the disturbance into a clearly defined **layer** *or* **series of layers** (*if they could but be seen*), *and an indefinitely thin portion of such a layer moves up and down or to and fro as a whole, and is called a wave front.*

The direction of progression of a water wave is at right angles to its front. **The direction of progression of a sound wave or light wave is at right angles to its front.***

---

* When the medium through which the wave passes has different properties in different directions, the direction of progression may not be at right angles to the front. Thus, in a substance like wood which has a grained structure, a sound wave does not in general progress in a direction at right angles to its front, in a crystal like Iceland spar a light wave does not in general progress in a direction at right angles to the front.

A wave which has a plane wave front is called a *plane wave.* A wave which has a spherical wave front is called a *spherical wave.*

14. Huygens' principle. Let AB, Fig. 4, be the instantaneous position of a wave which has come from a disturbance at C. The disturbance which is produced later at the point p when the wave reaches that point is of course to be thought of as having come originally from C; *it may be, however, considered to have come from the disturbance which constitutes the wave AB.* In this case, each point of the wave AB is to be considered as a center of disturbance from which a spherical wave emanates. The waves which thus emanate from each point of a *primary wave* are called *secondary waves* or *wavelets*. The actual disturbance produced at p is the resultant of the effects of all these wavelets.

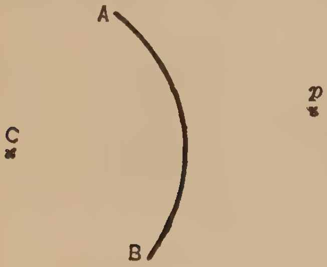

Fig. 4.

15. Huygens' construction for wave front. Let the line AB, Fig. 5, be a front in a wave which is traveling towards $A'B'$, and let it be required to find the wave front after a time has elapsed during which the wave has traveled a distance r. Describe circles (spheres) of radius r about each point of the wave front AB. The envelope $A'B'$ of these circles (spheres) is the required wave front. These spheres are the secondary wavelets described in the previous article.*

16. The ray of light. Consider a succession of waves

---

* A full discussion of Huygens' principle and of Huygens' construction for a wave front, together with Fresnel's improvement thereon, is to be found in Chapter III of Drude's *Theory of Optics*, translated by Mann and Millikan (New York, 1902).

WW, Fig. 6, traveling out from a center of disturbance *O*. These waves progress at each point in a direction at right angles to the wave front, and the lines *rrr* along which the wave disturbance travels are called *rays*. A bundle of rays drawn from the various points of and at right angles to a portion of a wave front constitutes what is called a *beam* or *pencil of rays*. For example, *W* in Fig. 7*a* represents a plane portion of a wave front, and the bundle of parallel lines *PP* is a pencil of parallel rays.

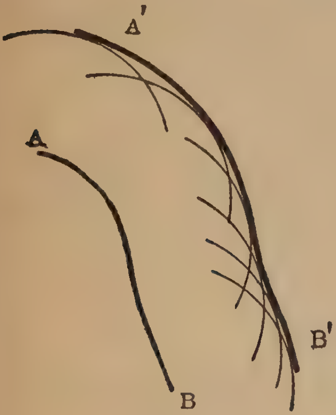

Fig. 5.

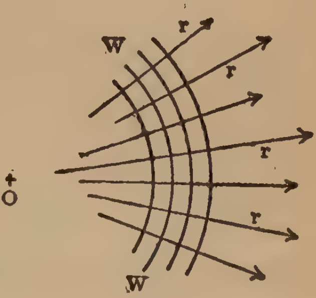

Fig. 6.

The conception of the ray carries with it the idea that a wave disturbance is propagated in straight lines, that a wave disturbance will not, for example, bend round an obstacle into the region behind the obstacle. Water waves do, however, bend round an obstacle and so also do sound waves, inasmuch as it is a common experience that a sound can be heard around a corner. The familiar phenomena of shadows* and the fact that one cannot see around a corner seem to indicate that light travels in straight lines. As a matter of fact, however, light bends round a corner in the same way that sound does but to an extent that is, under ordinary conditions, scarcely noticeable. This bending of light and sound around a corner is called *diffraction.*

---

* The phenomena of shadows are briefly discussed on pages 3 to 5 of Edser's *Light for Students*, Macmillan and Co., 1902.

17. Homocentric pencil of rays. When a portion of a wave front is a sector of a spherical surface (the entire wave front may not be a sphere) the rays from the portion intersect at the center of the sphere, the pencil of rays is said to be *stigmatic* or *homocentric,* and the center of the sphere is called the *focal point* of the pencil.

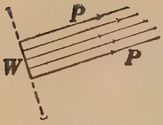

Fig. 7*a*.

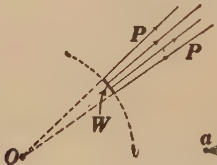

Fig. 7*b*.

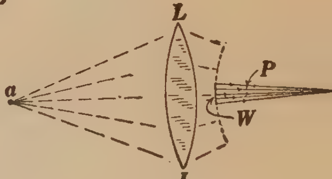

Fig. 7*c*.

In Fig. 7*b* the portion *W* of the wave front is spherical, and the wave is traveling out from the point *O*. In this case the pencil of rays *PP* is called a *divergent homocentric pencil.*

In Fig. 7*c* the portion *W* of the wave front is spherical, and the wave is traveling towards the point *b*. In this case the pencil of rays *P* is called a *convergent homocentric pencil.*

18. Astigmatic pencil of rays. Imagine a portion of a wave front shaped like a blow-out patch for an automobile tire. Such a portion of a wave front is represented by *WW* in Figs. 8 and 9; it is more sharply curved in one direction than in the direction at right angles thereto. The pencil of rays drawn from such a portion of a wave front constitutes what is called an *astigmatic pencil*. The rays in an astigmatic pencil do not meet at a point, but every ray passes through the *focal line* *CC* and through the focal line *DD*.

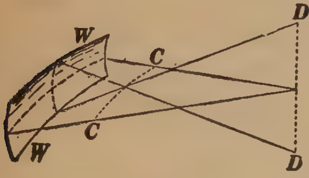

Fig. 8.

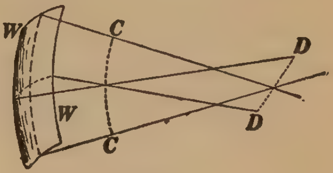

Fig. 9.

Example. A narrow pencil of rays from the point *a* in Fig. 10 passes obliquely through a lens *LL* as shown, and the emergent rays constitute an astigmatic pencil which is focussed first along the line *C* (which is at right angles to the plane of the paper), and then focussed along the line *DD*.

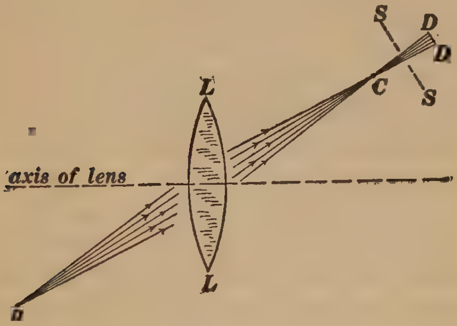

Fig. 10.

If the emergent rays in Fig. 10 are allowed to fall on a cardboard screen *SS* a short bright line will be seen on the screen if the screen passes through *C* or through *DD*. If the screen is between *C* and *DD*, as shown in Fig. 10, a blurred spot will be seen on the screen. The light from *a* which passes obliquely through the lens *LL* cannot be focussed at a point on the screen. The nearest approach to a point focus is obtained when the screen *SS* is midway between *C* and *DD*, in which case the blurred spot is a small circle which is called the *circle of least confusion*.

LEADING QUESTIONS

L. Q. 4. Draw a diagram showing a snap-shot of a stretched wire along which an isolated wave-pulse is traveling.

L. Q. 5. Draw a diagram showing a snap-shot of a stretched wire along which a wave-train is traveling.

L. Q. 6. What kind of a disturbance produces an isolated wave-pulse? What kind of a disturbance produces a wave-train?

L. Q. 7. Consider a train of waves in the form of a series of long parallel ridges traveling over a water surface. Is it correct to speak of *the* front *of* one of these waves, or to speak of *a* front *in* or *on* one of these waves? Why?

L. Q. 8. Imagine a series of waves traveling out from a point on a water surface where drops of water are falling. It is correct to speak of *rays* in connection with these waves. What is meant by a *ray* in this case? What would a *pencil of rays* be in this case?

L. Q. 9. What do the terms *ray* and *pencil of rays* mean in connection with a series of sound waves? In connection with a series of light waves?

L. Q. 10. What is the shape of the wave-fronts in a pencil of parallel rays? In a homocentric pencil of rays? In an astigmatic pencil of rays?

PROBLEMS

4. What is the wave-length of the sound waves in air which goes out from a tuning-fork which makes 500 complete vibrations per second? The wave-length of yellow light is 59 millionths of a centimeter in air. What is the wave-length of yellow light in glass in which the velocity of the yellow light is 2/3 as great as its velocity in air? How many yellow-light waves would pass any given point per second?

Note. The number of waves of a wave-train which pass a given point per second is called the frequency of the wave-train.

5. A listener rides at a velocity of 90 feet per second past a whistle whose frequency is 1000 complete vibrations per second. At what frequency do the sound waves from the whistle strike the listener's ear, (*a*) while he is approaching the whistle, and (*b*) while he is receding from the whistle?

6. At what velocity would a star have to be moving towards the earth in order that light of wave-length $58{\cdot}9615 \times 10^{-6}$ cm. (as emitted by the star at rest) would have a wave-length of $58{\cdot}7846 \times 10^{-7}$, as seen by an observer on the earth. Assume the earth to be standing still. Ans. 100 kilometers per second.

CHAPTER III

PHOTOMETRY AND ILLUMINATION

19. Measure of opening of a cone; spherical angle. Consider a given cone. Imagine a sphere of radius r with its center at the apex of the cone. Let A be the area of the portion of the spherical surface which is inside of the cone. *The ratio* A/r^2 *is taken as the measure of the opening of the cone.* If A/r^2 is unity, the opening of the cone is unity, that is to say, we have what may be called a *unit cone*, and this unit cone is called a *ster-radian*. The opening of a cone is often called *spherical angle*, and the ster-radian is the unit of spherical angle. The entire area of a sphere is $A = 4\pi r^2$, and therefore the spherical angle which corresponds to the entire sphere is $4\pi r^2/r^2$, or 4π ster-radians.*

20. The candle-power; A unit of conical intensity. Light is radiated in straight lines and therefore all the light from a very small source (from an ideal point source) which lies inside of a given cone whose apex is at the source stays in the cone. Therefore the *amount-of-light-per-ster-radian* is an important quantity; this quantity is called *conical intensity*, and its unit is the *candle-power*.

For the sake of concreteness a narrow horizontal beam from a standard candle† may be taken as one candle power, see Fig. 11. This was at one time accepted as the actual definition of the candle-power unit, but the standard candle power in the United States is now embodied in a number of carefully preserved tungsten-filament electric lamps which are to be operated at a specified voltage. These lamps are kept at the Bureau of Standards, and the Bureau supplies duplicates of them at a small cost.

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* The entire circumference of a circle is $2\pi r$ and the circular or plane angle which corresponds to the entire circumference of a circle is $2\pi r/r$, or 2π radians.

† The British Standard Candle, now obsolete, was made of spermaceti and was supposed to burn 120 grains of sperm per hour.

The Director of the Bureau of Standards says, " It is recognized that this is, perhaps, a temporary expedient, and that a reproducible primary standard should eventually be agreed upon. Up to the present time, however, no standard lamp has been devised which can be reproduced with an accuracy approaching that with which the candle-power unit can be preserved by means of electric lamps."

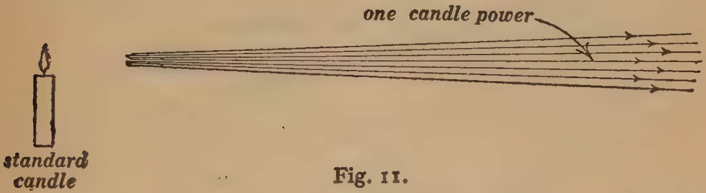

Fig. 11.

21. The lumen; a unit of light flux. The candle power is *not* a unit of light quantity (or light flux), it is a unit of conical intensity. Imagine a divergent beam of light whose conical intensity is the same in all directions and equal to one candle power. *The amount of light flux in one ster-radian of this beam is called a* **lumen,** that is to say, the candle power is one lumen of light flux per ster-radian.

Imagine a lamp which gives off light equally in all directions, and suppose the conical intensity of the emitted light were C candle power (C lumens per ster-radian). The total light flux emitted by such a lamp would be $4\pi C$ lumens, because the spherical angle all the way around the lamp is 4π ster-radians and there would be C lumens in each ster-radian.

22. Sectional intensity I of a beam of light. The law of inverse squares. Consider a narrow divergent beam of light as shown in Fig. 12. Let C be the candle power of the beam, and let $\triangle A$ be the sectional area of the beam at a place p which is r feet from the lamp. Then $\triangle A/r^2$ is the spherical angle of the beam and $C \times \triangle A/r^2$ is the amount ot light flux in the beam in lumens; and if we divide this by the sectional area $\triangle A$ of the beam at p we get the number of lumens per square foot of section at p. This result, lumens per square foot of section, is called the *sectional intensity*, I, of the beam at p. Therefore we have

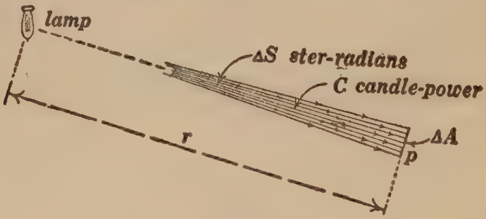

Fig. 12.

$$I = \frac{C}{r^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

which, being interpreted, means that the sectional intensity of a C-candle-power beam at a distance of r feet from the lamp is equal to C/r^2 lumens per square foot. This equation expresses what is called the "law of inverse squares," and it is the most important fundamental equation in photometry.

23. Intensity of Illumination of a Surface. A surface is said to be illuminated when light falls on it, and the intensity of illumination is expressed as so many *lumens per square foot.*

Case I. If a narrow divergent beam of light falls *perpendicularly* on a surface BB, as shown in Fig. 13, then the intensity of illumination of the illuminated part of the surface is, of course, equal to the sectional intensity I (lumens per square foot) of the divergent beam at the surface, as given by equation (i) of Art. 22, where C is the candle power (lumens per ster-radian) of the divergent beam.

Case II. If a narrow divergent beam of light falls *obliquely* on a surface BB, as shown in Fig. 14, then the intensity of illumination, I', of the illuminated part of the surface is $I' = I \cos \theta$, or, using the value of I from equation (i) of Art. 22, we have

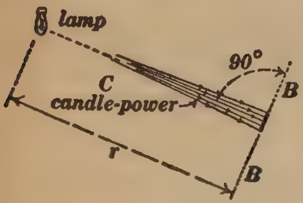

Fig. 13.

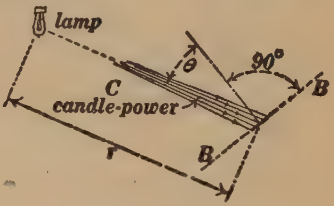

Fig. 14.

$$I' = \frac{C}{r^2} \cdot \cos \theta \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

where θ is the angle between the narrow divergent beam of light and the normal to the illuminated surface as shown in Fig. 14.

24. The Bunsen photometer. The most extensively used device for comparing the conical intensities (candle powers) of the light from two lamps is the *Bunsen photometer,* the essential features of which are shown in Fig 15. A screen is placed between the two lamps A and B as shown in Fig. 15, and the distances a and b are adjusted until the screen is equally illuminated on the two faces.* The intensity of illumination of the screen by lamp A is C/a^2, and the intensity of illumination of the screen by lamp B is C'/b^2, according to equation (i) of Art. 22. Therefore, since the two faces of the screen are equally illuminated, we have

$$\frac{C}{a^2}=\frac{C'}{b^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

from which either candle power can be calculated when the other is known and when a and b have been measured.

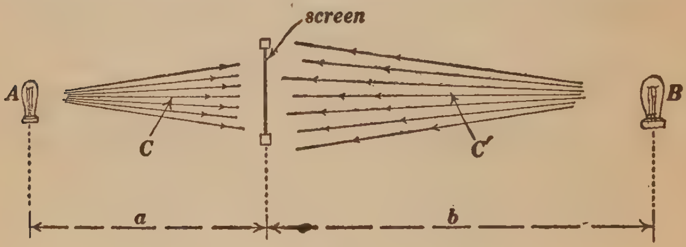

Fig. 15.

Note. If the two lamps are even slightly different in color the setting of the screen in Fig. 15 to give equal intensities of illumination on the two faces cannot be made accurately. This difficulty is to a large extent overcome by the use of the flicker photometer, which is described in Art. 111 of Chapter XI.

25. Calculation of Total Light Flux from a Lamp. Let the radii of the curve ccc in Fig. 16 represent the observed candle powers of the lamp O in the respective directions, and *let us suppose the lamp O to be completely symmetrical with respect to the axis DD* (or to be rotated rapidly about DD as an axis while the candle-power readings are being taken). Then the candle power Oa towards the middle of the narrow zone zz of the reference sphere may be taken as the mean candle-power over the zone, and if we multiply this candle power Oa by the spherical angle $2\pi h/r$ of this zone we get the number of lumens crossing the zone. The reference sphere may be wholly subdivided into zones, and the light flux across each zone calculated in this way, whence the total light flux is known.

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* Special arrangements are made so that the two faces of the screen are seen side by side.

This somewhat tedious calculation may be greatly

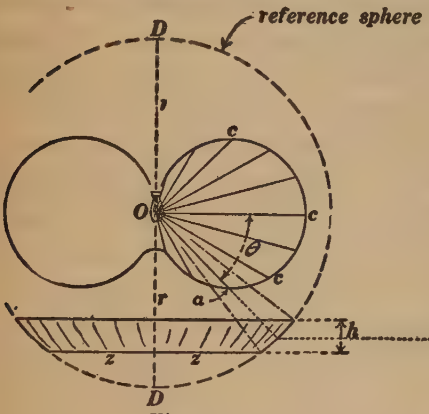

Fig. 16.

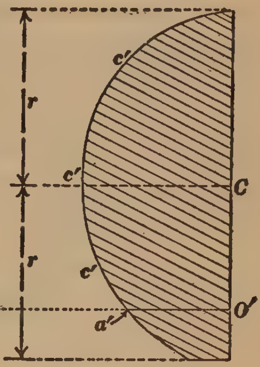

Fig. 17.

facilitated by a method due to Rousseau as follows: For each photometer reading Oa in Fig. 16 lay off CO', Fig. 17, equal to $r \sin \theta$, where r is a chosen length, and lay off $O'a'$ to represent the observed candle power, thus plotting the curve $c'c'c'$. Measure the shaded area in Fig. 17 by means of a planimeter and multiply this measured area in square inches by $2\pi s/r$ where s is the scale to which $O'a'$ represents candle power (candle power per inch) and r is the chosen length expressed in inches. The result is the total light flux in lumens emitted by the lamp.

Note 1. This rule for calculating the total number of lumens from the measured value of the shaded area in Fig. 17 is easily derived from the above discussion of Fig. 16.

Note 2. The Ulbricht globe photometer* is a device which gives the total number of lumens emitted by a lamp from a single reading; but the Ulbricht globe photometer must be standardized by taking its reading for a selected lamp for which the total emission in lumens has been determined as above explained, and when thus standardized the Ulbricht globe photometer gives accurate results only for lamps which are nearly similar in design to the selected lamp.

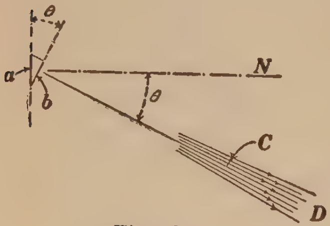

Fig. 18.

26. Brightness of a surface. Let *a*, Fig. 18, be an illuminated (or a luminous) surface, and let *C* be the candle power of the narrow beam which is emitted by *a* in the direction *aD*. Then *the brightness of a as seen from D is, by definition, equal to C/b*, where $b\ (= a \cos \theta)$ is the projection of the area *a* on a plane at right angles to *aD*. Therefore, representing the brightness of *a* as seen from *D* by *B*, we have

$$B = C/(a \cos \theta) \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

or

$$C = Ba \cos \theta \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

The line *aN* in Fig. 18 is at right angles to *a*.

Note. The brightness of a surface is not at all the same thing as the intensity of illumination of a surface. Thus a sheet of white paper may be fairly bright when not intensely illuminated, whereas a sheet of black velvet will not be very bright even when it is intensely illuminated.

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* See *Johns Hopkins Lectures on Illuminating Engineering*, Vol. I, pages 481–485, Baltimore, 1911.

Remark. When applied to the luminous surface of a lamp the term *intrinsic brightness* is frequently used instead of the simpler, and, as above defined, exactly equivalent term *brightness*. Thus the intrinsic brightness of an ordinary kerosene lamp flame seen flatwise is about 3 candle power per square cm., and the intrinsic brightness of an ordinary tungsten lamp filament is about 200 candle power per square cm.

27. Lambert's law. Let a in Fig. 18 be a non-glossy surface like the surface of a piece of blotting-paper; with given intensity of illumination *the brightness B of such a surface is the same from whatever direction it be seen.* That is to say, the quantity B in equations (i) and (ii) of Art. 26 is the same in value for all values of the angle θ, and an interesting consequence of this fact is that the candle power C in Fig. 18 is proportional to cos θ if the surface a is a non-glossy surface. This is in accordance with equation (ii) of Art. 26, for, when a is a non-glossy surface the quantity B is constant.

28. Proposition. Let B be the brightness in candle power per square foot of a flat, non-glossy surface whose area is a square feet. Then the number of lumens of light flux emitted by this surface is

$$L = \pi B a \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Proof. Let ΔA in Fig. 19 be the non-glossy surface (small). Imagine a sphere of radius r with its center at ΔA. This we will call the reference sphere for convenience. Consider the zone zz of this reference sphere. This zone extends entirely around the "pole" p of the sphere, the area of the zone is $2\pi r \sin\theta \times r \cdot d\theta$, the spherical angle subtended by this zone is found by dividing its area by r^2, and the number of lumens of light flux dL crossing the zone zz is found by multiplying the spherical angle of the zone by the candle power

$C = B\cdot\triangle A\cdot\cos\theta$ of the light which reaches the zone [see equation (ii) of Art. 26]. Therefore

$$dL = 2\pi B\cdot\triangle A\cdot\sin\theta\cos\theta\cdot d\theta$$

and to find the total number of lumens emitted by $\triangle A$ we must integrate this expression from $\theta = 0$ to $\theta = \pi/2$, which gives $L = \pi B\cdot\triangle A$. Q.E.D.

29. Limitations of the law of inverse squares. Strictly speaking, the light from an *ideal point-source*, only, has a definite candle power (conical intensity) because it is only from a point-source that the rays lie in clearly defined cones; also the law of inverse squares is strictly true, only, for light which emanates from a point-source. Therefore in all of the preceding articles the lamp, or light source, is assumed to be small in size.

But lamps and light sources are not small, and yet the law of inverse squares is used in nearly all photometric calculations! It is important, therefore, to determine how large a lamp, or light source, may be without introducing an inaccuracy of more than about two-tenths of one per cent in the inverse square law, two-tenths of one per cent being about the limit of accuracy of photometric measurements.

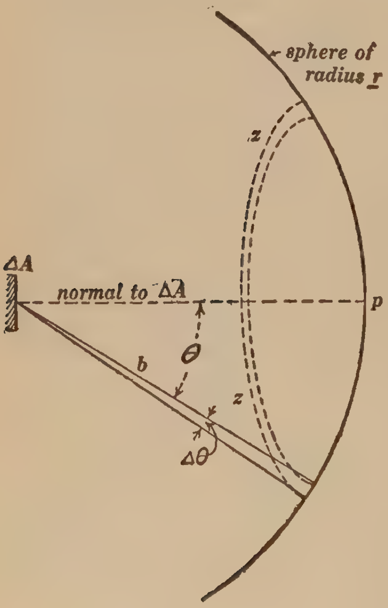

Fig. 19.

The heavy line *DD* in Fig. 20 represents a flat circular disk of illuminated blotting-paper which emits light in accordance with Lambert's law. The light from *DD* illuminates a small area at *p* which is at a distance *R* from the center of *DD* as shown. Let *J* be the intensity of illumination of the small area at *p* by the light from *DD*, and let *I* be the intensity of illumination of the small area at *p which would be produced by a very small disk put in place of DD on the supposition that the small disk gives off the same number of lumens as the large disk DD*. Then $J = I$ when $R =$ infinity, and as *R* decreases *I* increases in accordance with the inverse-square law, as shown by the heavy-faced numbers in the accompanying table, but as *R* decreases *J* increases less rapidly than *I*, as shown by the light-faced numbers in the table.

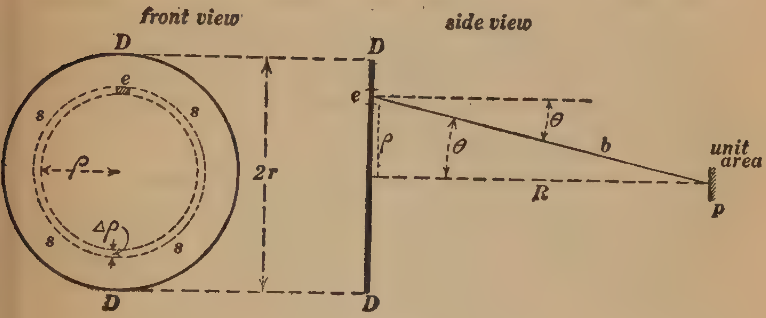

Fig. 20.

TABLE

Intensities of illumination of the small area at p in Fig. 20.

Values of *R* in feet......	2½	5	10	20	40	80
Values of *J* due to disk *DD* one foot in diam. .	984·6	253·5	63·84	15·99	3·9992	0·99996
Values of *I* due to very small disk in place of *DD*	**1024**	**256**	**64**	**16**	**4**	**1**

According to the accompanying table, the inaccuracy of the law of inverse squares for the flat paper disk one foot in diameter is less than 0·2 per cent when the distance R is greater than 10 feet (when R ranges from 10 feet to infinity), and this inaccuracy is greater for a flat paper disk than for most lamps. Therefore *it is safe to use the law of inverse for all cases in which the maximum dimension of the luminous source does not exceed one-tenth of the distance of the lamp, or source, from the photometer screen or other illuminated surface.*

30. Calculation of the above table. Figure 20 shows a blotting-paper disk DD of brightness B, and it is required to find the sectional intensity at the point p of the light from DD, or, in other words, it is required to find the number of lumens of light falling on unit area at p. The radius of the disk is r and the distance of p from the middle of the disk is R.

Consider the beam of light which reaches p from the small surface-element e of the circular strip ss of the paper disk. The conical intensity (candle power) of this beam is $Be \cos \theta$, according to Art. 27, and the sectional intensity S of this beam at p is found by dividing this candle power by the square of the distance ep, which is $\rho^2 + R^2$. Therefore $S = Be \cos \theta/(\rho^2 + R^2)$. Now the projection of the unit area at p on a plane at right angles to ep is $1 \times \cos \theta$, and if we multiply this projected area by S we get the number of lumens dL falling on the unit of area. Therefore using $\rho \cdot d\phi d\rho$ for e, where $d\phi$ is the angle subtended by e at the center of the disk, and putting $R/(R^2 + \rho^2)^{1/2}$ for $\cos \theta$, we have

$$dL = \frac{BR\rho}{(R^2 + \rho^2)^2} \cdot d\Phi d\rho$$

and, integrating with respect to Φ from $\Phi = 0$ to $\Phi = 2\pi$ and with respect to ρ from $\rho = 0$ to $\rho = r$, we get

$$J = L = \frac{\pi B r^2}{R^2 + r^2}$$

from which the values of J in the above table were calculated.

The numerator $\pi B r^2$ ($B \times$ area of disk) when multiplied by π gives the total number of lumens emitted by the disk, according to Art. 28, and this is to be the same for the large disk and for the very small disk. Therefore the values of I in the accompanying table were calculated from $I = k/R^2$, where k is chosen so as to make $k/80^2$ equal to unity; and, using the same value of k, the values of J were calculated from $J = k/(R^2 + r^2)$.

31. The illumination of a room.* A room may be said to be well lighted when the eye is easily able to distinguish the various objects in the room in minute detail of perception. This completeness of visual perception depends upon three conditions : namely, (*a*) a sufficient brightness of illumination, (*b*) a proper location of the light sources so to bring out that combination of soft shadows which is so essential to the perception of form, and (*c*) a proper composition† of the light so as to bring out those physical differences in objects which the eye perceives as variations of color.

(*a*) The necessity of having a sufficient intensity of illumination is, of course, known to everyone. The ability to perceive fineness of detail (called *visual acuity*) depends chiefly upon intensity of illumination.

Visual acuity is always measured in an arbitrary way ; for example, one may measure visual acuity as the distance from one's eye at which clear black print of a chosen size may be read, and the dependence of visual acuity upon intensity of illumination may be determined by finding the distance at which the given type can be read for different intensities of illumination. In this way it is found that visual acuity is very low when the intensity of illumination is one or two-tenths of a lumen per square foot, it increases rapidly up to one or two lumens per square foot, and then it increases slowly and reaches a maximum at about eight or ten lumens per square foot.

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* An extremely interesting discussion of the conditions which determine visual perception is given by Helmholtz in his popular lecture on *The Relation of Optics to Painting*, which is translated (by E. Atkinson) in the second series of Helmholtz's *Popular Lectures*, Longmans, Green & Co., 1903. Everyone who is concerned with the practical problems of illumination should read this lecture. Helmholtz's *Popular Lectures* are published in German under the title *Vorträge und Reden*, 2 volumes, Braunschweig, Vieweg und Sohn, 1884.

Three lectures in Helmholtz's first series (translated by Dr. Pye-Smith ; Longmans, Green & Co., 1873), *On the Theory of Vision*, also have a bearing upon the important practical subject of illumination.

See also the lectures by Percy W. Cobb and by Robt. M. Yerkes, pages 525–604, Vol. II, *Johns Hopkins University Lectures on Illuminating Engineering*, Baltimore, 1911.

The practical method for choosing and locating the lamps for illuminating a room is explained on pages 164–170 of W. S. Franklin's *Elements of Electrical Engineering*, Vol. II.

† The composition of light refers to the relative intensities of the various wave-lengths which are present in the light. See Chapter VIII.

(*b*) A room may be sufficiently illuminated by a single arc lamp, but such illumination is unsatisfactory, even when the eye is shaded from the direct light of the lamp, because the excessive harshness of the shadows renders the perception of form almost impossible. The light from a single brilliant lamp is always softened, however, by the reflection from the walls and ceiling of a room.

The second condition is not important where purely flat-surface vision is required as in a draughting room, where indeed it is important to eliminate all shadows on the sheet of drawing paper.

(*c*) The importance of the third condition is evident when one attempts to distinguish delicate colors by ordinary lamp light. Thus the light of an ordinary kerosene lamp is very deficient in the short wave-lengths (blue and violet), and a deep blue or violet piece of cloth appears almost black by kerosene lamp light. False color values are produced in a very striking way by the light from a mercury-vapor lamp on account of the almost complete absence of the longer wave-lengths (red) in the light from this lamp. The most striking illustration of false color values, however, may be obtained by illuminating a batch of brilliantly colored worsteds by the light from a sodium flame in a room from which all white light is excluded. All differences of tint disappear under these conditions, and a given piece of worsted merely appears to be light or dark according as it is able or unable to reflect the yellow light of the sodium flame.

32. Glare. The presence of excessively brilliant lamps or excessively brilliant patches of light in a field of vision greatly hinders visual perception. The eye adapts itself automatically to the brightest lights in the field of view, and all perception of detail in the shadows is lost. This effect is called *glare*, and it is especially marked when the field of vision includes a bright unshaded lamp.

The explanation of glare is as follows: In the first place, a beam of light entering the eye from a bright source illuminates the whole interior of the eye just as a beam of sunlight entering a window illuminates a room. This diffused light in the eye illuminates and excites the entire retina, including those portions where the images of the deeper shadows fall, and thereby tends to obliterate all detail of perception. In the second place, the portions of the retina upon which the brilliant light falls become greatly reduced in sensitiveness by fatigue, the continual wandering of the eye brings the image of a dark region upon this fatigued portion of the retina, and the result is almost total blindness like that produced when one looks out of a window and then turns towards a dark corner of a room. In the third place, the pupils of the eyes contract greatly when there is a bright light in the field of vision, and this contraction lessens the effective brightness not only of the bright portions of the field, but also of the deep shadows; but the deep shadows are already insufficiently illuminated and the contraction of the pupils of the eyes tends to make them (the shadows) appear like black patches entirely devoid of detail.

It is very important, in arranging for the illumination of a room, to place the lamps outside of the field of vision if possible, so that no light can enter the eye directly from the lamps. The excessive discomfort that is produced by the glare of improperly located lamps, such, for example, as the exposed footlights of a poorly arranged stage, is due not only to the physical pain that is associated with long-continued looking at a bright light, but more especially to the incessant effort of trying to peer into the dark region beyond.

When a lamp cannot be removed from the field of vision the bad effects of glare may be greatly reduced by enlarging the luminous surface of the lamp by means of a translucent globe or shade.

LEADING QUESTIONS

L. Q. 11. Define the radian of plane angle. How many radians of plane angle correspond to the entire circumference of a circle? Why? Define the ster-radian of spherical angle. How many ster-radians of spherical angle correspond to the entire surface of a sphere? Why?

L. Q. 12. Does the conical intensity (candle power) of a divergent beam become less and less at greater and greater distances from a lamp? If not, why not?

L. Q. 13. What is meant by one lumen of light flux?

L. Q. 14. What is meant by the sectional intensity of a beam of light at a given place, and in terms of what units is sectional intensity expressed?

L. Q. 15. An intensity of illumination of one lumen per square foot is often called a "foot-candle." What is the objection to this term?

L. Q. 16*a*. Derive the expression $2\pi h/r$ for the spherical angle of the zone zz in Fig. 16, where h is the height of the zone as shown in Fig. 16 and r is the radius of the reference sphere.

L. Q. 16*b*. Prove that the total light flux emitted by the lamp in Fig. 16 is equal to $2\pi s/r$ times the shaded area in Fig. 17, using the notation of Art. 25.

L. Q. 17. Lambert's law as usually stated is based on equation (i) of Art 26. Give a verbal statement of the meaning of this equation, beginning thus: The candle power of the light beam emitted in a given direction by a small non-glossy surface is proportional to the cosine of the angle, etc. etc.

L. Q. 18. Draw a curve like *ccc*, Fig. 16, the radii of which represent the candle powers in the different directions of the light emitted by a small non-glossy illuminated surface at *O*, the plane of the illuminated surface being at right angles to the plane of the paper in Fig. 16.

L. Q. 19. State and explain the meaning of each of the three conditions which must be met in good illumination of a room.

L. Q. 20. What are the three effects which enter into glare?

PROBLEMS

7. The horizontal beam of light from an ordinary paraffine candle (see Fig. 11) represents the flow of about 450 ergs of radiant energy per second across an area of 1 square centimeter at a distance of 1 meter from the candle, and of this total flow of radiant energy only about 9·3 ergs per second is light (radiant heat whose wave length lies between 39 and 75 millionths of a centimeter, see Chapter VIII). A paraffine candle burns about 8 grams of paraffine per hour, and the heat of combustion of paraffine is 10,000 calories per gram. Let us assume for the sake of simplicity that the candle flame radiates equally in all directions and that the emitted light has one candle power in all directions. (*a*) Find the fraction of the heat of combustion which is radiated by the candle flame. What becomes of the remainder? (*b*) Find the fraction of the total heat of combustion which is radiated by the candle flame as light. (*c*) Find the number of watts represented by each lumen of light flux emitted by the candle flame. Ans. (*a*) 6·06 per cent; (*b*) 1/8 of one per cent; (*c*) 0·0093 watt.

8. An ordinary 100-watt tungsten-filament electric lamp gives off about 800 lumens of light. Taking 0·0093 watt as the actual power equivalent of each lumen, find the fraction of the power delivered to the lamp which is emitted as light? What becomes of the remainder? Ans. $7\frac{1}{2}$ per cent.

9. The conical intensity of the light from a lamp is 50 lumens per ster-radian (50 candle power) in all directions. The lamp is 4 feet from the center of a circular hole in a wall, the diameter of the hole is 6 feet, and the wall is perpendicular to the line drawn from the lamp to the center of the hole. Find the amount of light flux passing through the hole. Ans. 62·8 lumens.

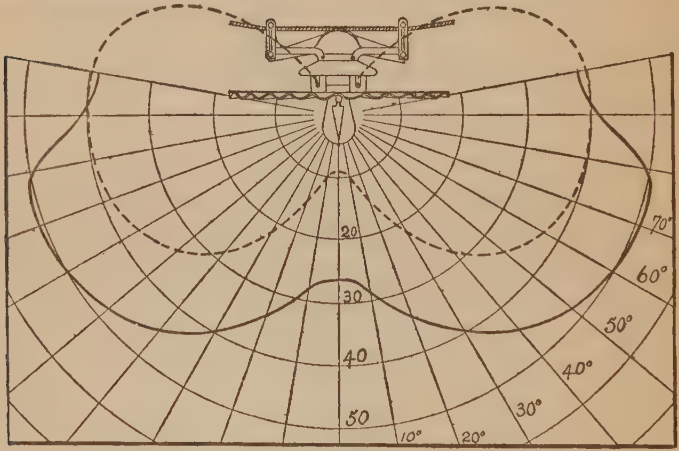

Fig. 21.

10. The lamp of the previous problem is placed at the center of a circular band which is 4 feet in diameter and 1 foot wide. How much light flux falls on the band? Ans. 152 lumens.

11. A lamp which gives off light uniformly in all directions is 6 feet from the center of a flat circular disk which is 2 feet in diameter, and the disk is perpendicular to the line drawn from the lamp to the center of the disk. Find the intensity of illumination of the edge of the disk expressed as a fraction of the intensity of illumination of the center of the disk. Ans. 0·96.

12. A lamp is 5 feet above one end of a level table and the table is 10 feet long. The downward beam from the lamp has a conical intensity of 50 candle power, and the beam which illuminates the distant end of the table has a conical intensity of 30 candle power. Find the intensity of illumination of each end of the table. Ans. (*a*) 2 lumens per square foot; (*b*) 0·107 lumen per square foot.

13. Figure 21 is a side view of a common type of street lamp. The radii of the dotted curve represent the candle powers (in the respective directions) of the unshaded lamp, and the radii of the full-line curve represent the candle powers (in the respective directions) of the lamp with its shade. Two such lamps are 16 feet above a level side-walk and 100 feet apart. Find the intensity of illumination of a spot on the side-walk which is 30 feet from one lamp and 70 feet from the other lamp. Ans. 0·023 lumen per square foot.

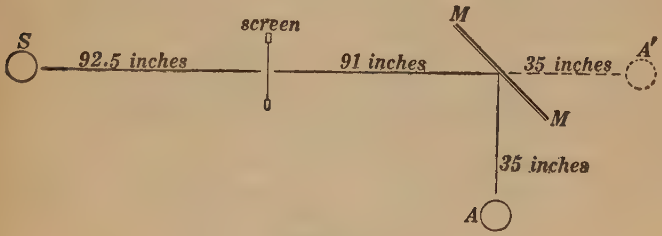

Fig. 22.

14. A certain photographic lens gives a good photograph with an exposure of 0·02 second when the sun is 75° above the horizon. What exposure would be required when the sun is 5° above the horizon, if it were not for the greatly increased absorption of the sun's rays by the atmosphere when the sun is low? Ans. 0·22 second.

15. Figure 22 shows a large plane mirror *MM* reflecting the light from a lamp *A* to the screen of a Bunsen photometer. The candle power of lamp *A* in the upward direction in Fig. 22 has been previously measured and found to be 35·3, the candle power of the standard lamp *S* in the direction towards the photometer screen is 16·8, and when the screen is equally illuminated on both sides the distances are as shown in the figure. Find the factor by which the apparent candle power of any lamp (when measured by the light reflected from MM as in Fig. 22) must be multiplied to correct for the imperfect reflection by the mirror. Ans. 1·13.

Note. Consider that the right side of the screen in Fig. 22 is illuminated by light from a lamp at A', where A' is the image of A in the mirror.

16. Three plane mirrors, which are represented by the heavy black lines in Fig. 23, are mounted in a structure which can be rotated about the axis of the photometer so the light emitted by the lamp L in any direction in a plane at right angles to the plane of the paper in Fig. 23 can be reflected along the axis of a Bunsen photometer to the photometer screen. Make a sketch of Fig. 23, and in this sketch locate the image a of the lamp in the first mirror; locate the image b (of a) in the second mirror; locate the image c (of b) in the third mirror; and show that the distance of c from the photometer screen is $f + g + h + i$.

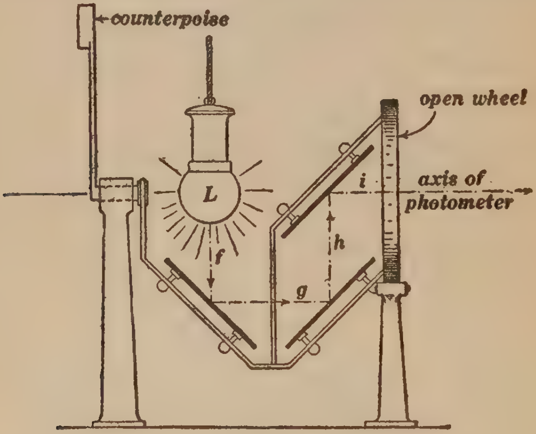

Fig. 23.

17. The candle power of the lamp L in the direction towards the reader in Fig. 23 has been measured (without the use of the mirrors, of course) and found to be 329. The mirror-structure in Fig. 23 is then turned so that the line *f* is horizontal and towards the reader, and the photometer screen which is used in Fig. 23 is moved until it is equally illuminated on both sides. The standard lamp used has a candle power of 17 and it is 30 inches from the photometer screen (far to the right in Fig. 23). The distance *f*, *g*, *h*, and *i* are 20, 20, 20, and 56 inches respectively. Find the factor by which any observed candle power must be multiplied to correct for the imperfect reflection of the three mirrors in Fig. 23. Ans. 1·30.

18. A flat circular disk of blotting paper 6 inches in diameter is placed at a distance of 2 feet from a lamp which gives a 2000-candle-power beam towards the disk. The plane of the disk is at right angles to the line drawn from the lamp to the center of the disk, and the candle power of the beam emitted by the disk in a direction inclined 60° to the normal to the disk is found by test to be 14. Assuming Lambert's law and treating the 6-inch disk as a point source, find the fraction of the incident light which is reflected (diffusely reflected) by the disk. Ans. 0·90.

CHAPTER IV

REGULAR REFLECTION AND REFRACTION

33. Definition of index of refraction of a substance. The velocity of light is different in different substances. Let V be the velocity in vacuum, let A be the velocity in air at 0° C. and 760 mm. pressure, and let S be the velocity in a given transparent substance. Then the ratio V/S is called the *index of refraction of the substance referred to vacuum,* and the ratio A/S is called the *index of refraction of the substance referred to air.* In all that follows the ratio A/S is represented by the letter μ.

The velocity of light in a vacuum is the same for all wave lengths (colors), but the velocity of light in every known substance varies with the wave length (color). This matter is discussed in Chapter VIII. In this chapter the index of refraction is thought of as having a definite single value for a given substance, that is to say, this chapter applies in all strictness to light of one wavelength (color).

34. Application of Huygens' principle to regular reflection and refraction. In Figs. 4 and 5 of Chapter II AB is the position of an advancing wave front at a given instant, and the principle enunciated by Huygens is that each point of this wave front may be considered to be a center of disturbance from which secondary waves or wavelets pass out, and, of course, all such wavelets start out from AB simultaneously. In the application of Huygens' principle to reflection and refraction the principle must be modified, *each point of the reflecting and refracting surface must be considered as a center of disturbance when the advancing wave reaches that point, and, evidently, the secondary waves or wavelets do not start out from all points of the reflecting and refracting surface simultaneously.*

A light wave (wave front) *WW* comes from a light source *O* and approaches a polished glass surface *AB* as shown in Fig. 24, and *YYY* is the position the wave would have reached at a certain instant *y* if it had not encountered the glass. As a matter of fact, however, the single wave *WW* breaks up into two waves, one of which is thrown backwards or *reflected* by the glass surface, and the other enters the glass and is said to be *refracted*. *It is required to find the position of the reflected wave and the position of the refracted wave at the instant y.*

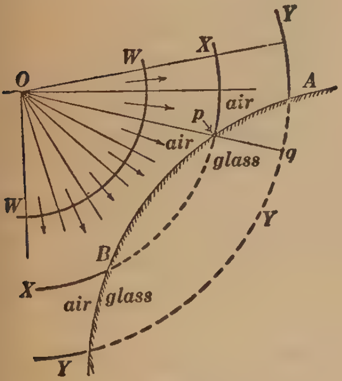

Fig. 24.

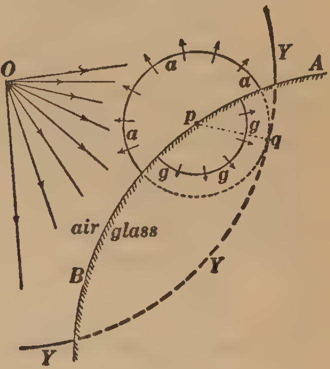

Fig. 25.

Consider the state of affairs at an instant *x*, a little earlier than the instant *y*. At this instant *x* the wave passes through the point *p* on the glass surface, the point *p* is therefore a center of disturbance at the instant *x*, and a Huygens' wavelet starts out from p at this instant. At the later instant y, the wavelet from p has had time to travel the distance pq in air [or the distance $\frac{1(pq)}{\mu}$ in glass]. Therefore (a) the portion of the wavelet from p *which is in air* is a hemisphere of which the radius is equal to pq, and this hemisphere is represented by the semicircle *aaa* in Fig. 25, and (b) the portion of the wavelet from p *which is in glass* is a hemisphere of which the radius is equal to $\frac{1}{\mu} \times (pq)$, and this hemisphere is represented by the semicircle *ggg* in Fig. 25.

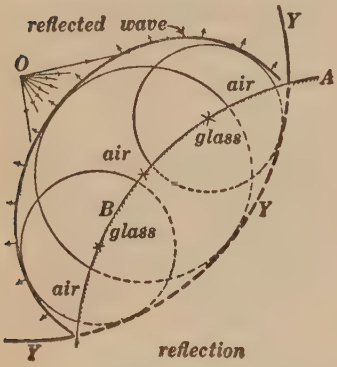

Fig. 26.

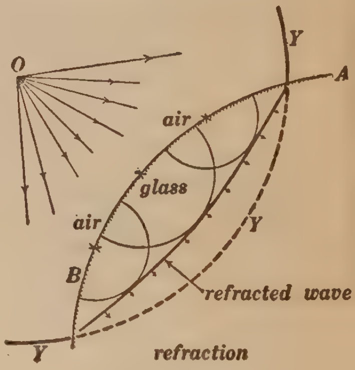

Fig. 27.

Therefore if we draw a system of circles (spheres) tangent to YYY and with their centers on AB, the envelope of these circles (spheres) will be the *reflected wave* as shown in Fig. 26; and if we replace each of the circles in Fig. 26 by a circle of which the radius is $1/\mu$ as great we will have the system of circles (spheres) in Fig. 27 of which the envelope is the *refracted wave.*

In all that follows the Huygens wavelets will be spoken of simply as circles as shown in the diagrams.

35. Reflection of a plane wave from a plane surface. Let WW in Fig. 28 represent a plane wave approaching a plane polished glass or metal surface AB, and let YY be the position this wave would have reached at a given instant y if it had not encountered AB. Then, according to Art. 34, aa is the position of the reflected wave at the instant y. The *incident rays* are perpendicular to WW, or YY, and the *reflected rays* are perpendicular to aa; and it is evident from the figure that the angle i (called the *angle of incidence*) is equal to the angle r (called the *angle of reflection*).

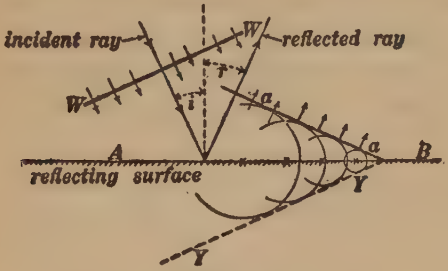

Fig. 28.

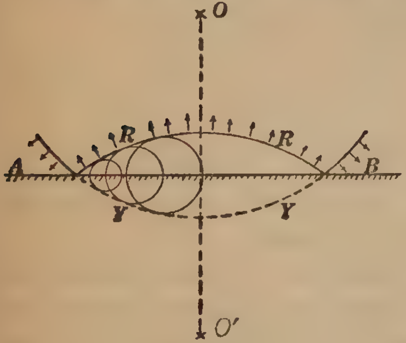

Fig. 29.

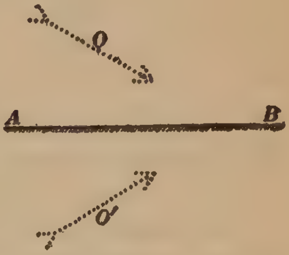

Fig. 30.

36. Reflection of a spherical wave from a plane surface.

Let O, Fig. 29, be a luminous point from which spherical waves emanate, let AB be a plane reflecting surface, and let YY be the position which a spherical wave from O would have reached at a given instant y if it had not encountered the reflecting surface AB. It is required to find the position of the reflected wave at the given instant y. About each point of the surface AB as a center describe a circle which touches YY. A few only of these circles are shown in the figure. The envelope RR of these circles is the reflected wave.

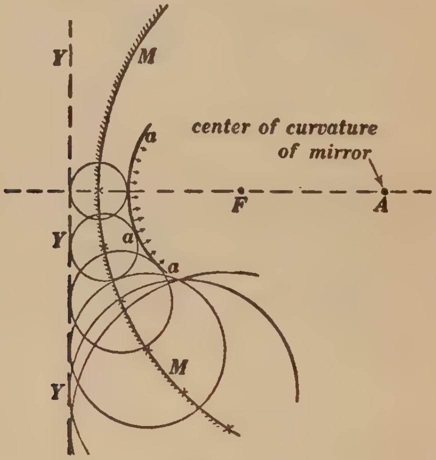

Fig. 31.

It is evident from the symmetry of the figure that the reflected wave RR is a sphere exactly like YY with its center O' so located that the line OO' is at right angles to AB and bisected thereby. After reflection from AB the light from O appears to come from O'.

37. Image of an object in a plane mirror. An ordinary object is a group of luminous points in so far as its light-giving properties are concerned. Consider such an object

O, Fig. 30. The light from the various points of this object, after reflection from a plane surface *AB*, or from a portion of such a surface, appears to come from a similar group of points *O′*. This group of fictitious luminous points *O′* is called the *image* of the object *O*.

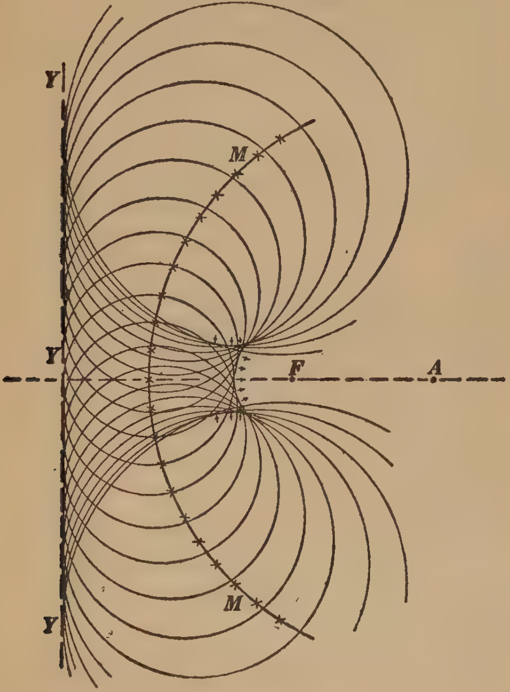

Fig. 32.

38. Reflection of a plane wave from a spherical surface. The reflection of a plane wave from a spherical surface is a complicated thing, too complicated to be discussed in detail in this text. The following discussion is based almost wholly on Figs. 31 and 32, and Fig. 32 is an accurate reproduction of a carefully made drawing. Reflection from a spherical mirror of small aperture is a comparatively simple thing, and this special case is discussed very briefly in Art. 39.

Let *MM*, Fig. 31, represent a polished spherical surface. A plane wave comes from the right, and *YY* is the position which this wave would have reached at a certain instant *y* if it had not encountered the mirror *MM*. Using the method of Art. 34, we find the position *aaa* of the reflected wave at the instant *y*. The reflected wave *aaa*, Fig. 31, is not spherical. Indeed Fig. 32 is a more extended drawing which shows the reflected wave *aa* with two cusps, and the heavy lines in Fig. 33 show two successive positions of the reflected wave. The small arrows in Figs. 31, 32, and 33 show the direction of travel of the reflected wave. The cusps of the reflected wave travel along the dotted lines *CC* in Fig. 33, and these dotted lines *CC** are called the *caustics* of the reflected wave.

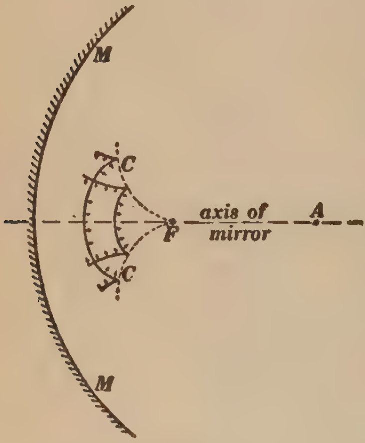

Fig. 33.

An interesting property of the caustic curve is shown in Fig. 34. The center of curvature of the small arc *AA* lies on the caustic surface at the point *c*. Indeed the pencil of rays corresponding to the small portion of the wave front *AA* is an astigmatic pencil One focal line of the pencil is at *c* (perpendicular to the plane of the paper) and the other focal line is *dd*.

---

* These lines, of course, represent a surface, for Fig. 33 represents a 3-dimensional arrangement. Rotate the figure about the axis of the mirror and the surface generated by the lines *CC* in Fig. 33 is the *caustic surface*.

39. Three simple cases of reflection from curved surfaces.

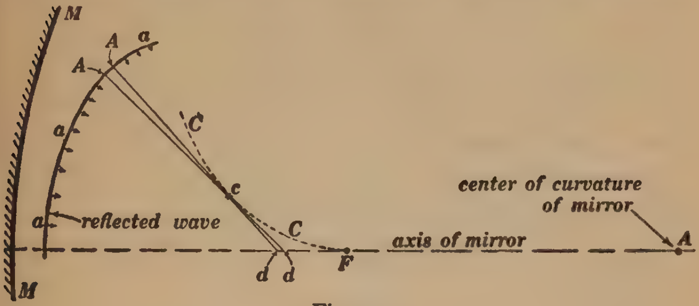

Fig. 34.

Case I. *The parabolic reflector.* The heavy line in Fig. 35 represents a mirror of which the reflecting surface is a paraboloid of revolution. A beam of parallel rays

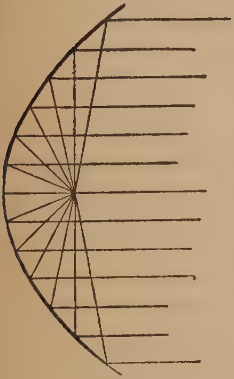

Fig. 35.

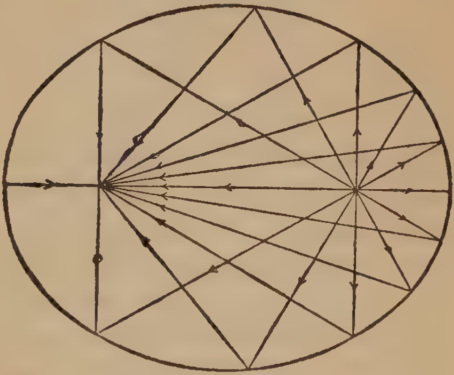

Fig. 36.

(parallel to the axis of the paraboloid) is concentrated at the focus of the paraboloid; or light emanating from the focus is reflected by the paraboloid as a beam of parallel rays (parallel to the axis of the paraboloid). The parabolic reflector is extensively used in automobile headlights and in searchlights.

Case II. *The ellipsoidal reflector.* The heavy line in Fig. 36 represents a hollow metal shell the interior (polished) surface of which is an ellipsoid of revolution (prolate ellipsoid). Light emanating from one focal point of the ellipsoid is concentrated at the other focal point. Vaulted ceilings sometimes approximate very closely to the ellipsoidal form, and a very faint sound at one focal point can be heard distinctly at the other focal point.

Case III. *The spherical mirror of small aperture.*

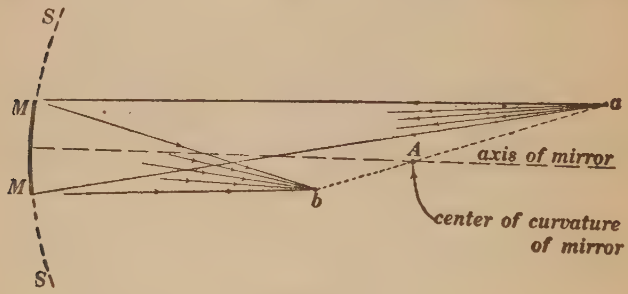

Fig. 37.

Figure 37 shows a mirror *MM*, the reflecting surface of which forms part of a sphere *SS* with its center at *A*. The diameter of *MM* is small in comparison with the radius of the sphere *SS*. Such a mirror is called *a spherical mirror of small aperture*, and the light from a luminous point *a* is *very nearly* concentrated at the point *b*. The theory of spherical mirrors of small aperture is very similar to the theory of lenses, and inasmuch as the theory of simple lenses is quite fully discussed in Chapter V, it is not worth while to discuss spherical mirrors.

40. Refraction of a plane wave at a plane surface. Consider a plane wave *WW*, Fig. 38, which is approaching the plane surface pO of a piece of glass, and let YY be the position which this wave would have reached at a certain instant y if it were not for the presence of the glass. Then, using Huygens' method as outlined in Art. 34, we find that the line Os' shows the actual position of the wave in the glass at the instant y, if the two distances ps and ps' are distances which would be traveled in equal times t in air and in glass, respectively; that is, if $ps = At$ and $ps' = Gt$, where A is the velocity of light in air and G is the velocity of light in the glass.

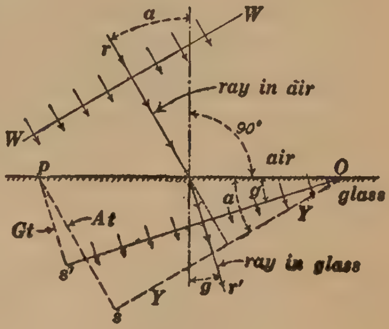

Fig. 38.

Therefore, from the right-angled triangles pOs and pOs', we have

$$\sin a = At/D \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

and

$$\sin g = Gt/D \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

where D is the distance pO. Consequently, dividing equation (i) by equation (ii), member by member, we get

$$\frac{\sin a}{\sin g} = \frac{A}{G} = \mu \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

where μ is the index of refraction of the glass referred to air as stated in Art. 33. The relation which is expressed by equation (iii) is called *Snell's law* from its discoverer.

The light is represented as passing from the air into glass in Fig. 38, and under these conditions the angle a is called the *angle of incidence* and the angle g is called the *angle of refraction*. If the light passes from the glass into the air, the two angles a and g are related in the same way [see equation (iii)], but in this case g is called the *angle of incidence* and a is called the *angle of refraction*. It is best to speak of a as the *angle in the air* and of g as the *angle in the glass*, and $\sin a/\sin g$ is equal to A/G irrespective of the direction of travel of the light.

41. The principle of Fermat. The law of regular reflection, namely, that the angle of incidence is equal to the angle of reflection (see Fig. 28), and the law of regular refraction as expressed by equation (iii) of Art. 40, may both be derived from a principle which was first enunciated by Fermat and which is called *Fermat's principle.*

As applied to refraction Fermat's principle is as follows: Consider that light is traveling from a point P in air to a point Q in glass, as shown in Fig. 39, and let p be the distance traveled in air and let q be the distance traveled in the glass. Then p/A is the time of travel in the air and q/G is the time of travel in the glass, where A is the velocity of the light in air and G is the velocity of the light in the glass. Then, according to Fermat, the path of the light (the positions of the rays in air and glass) is such that $p/A + q/G$ is a minimum. This condition gives equation (iii) of Art. 40.

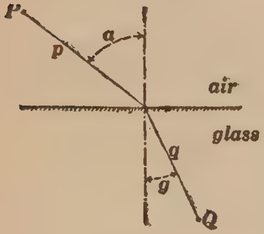

Fig. 39.

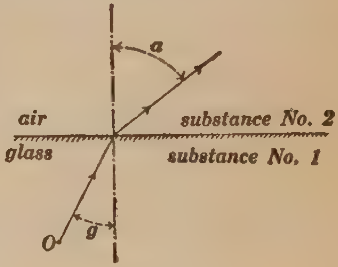

Fig. 40.

42. The phenomenon of total reflection. Figure 40 shows a ray of light passing from glass into air, and, in accordance with equation (iii) of Art. 40, we have

$$\sin a = \frac{A}{G} \sin g \ldots\ldots\ldots\ldots\ldots (i)$$

The angle g in Fig. 40 can evidently have any value whatever, and A/G is equal to 3/2 for ordinary glass and air. Therefore, according to equation (i), the sine of a would have to be greater than unity when the sine of g

is greater than 2/3! There is, of course, no such angle, and, as a matter of fact, none of the light passes out into the air in Fig. 40 when sin g is greater than 2/3; all of the light is reflected back into the glass as indicated in Fig. 41. The dotted line in Fig. 41 shows the critical position of the ray in the glass for which sin g is equal to G/A. All of the rays in Fig. 41 are reflected back into the glass to some extent, but the rays beyond the dotted line are completely reflected. This effect is called *the phenomenon of total reflection.*

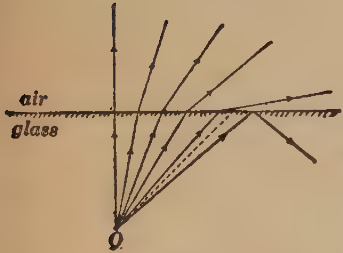

Fig. 41.

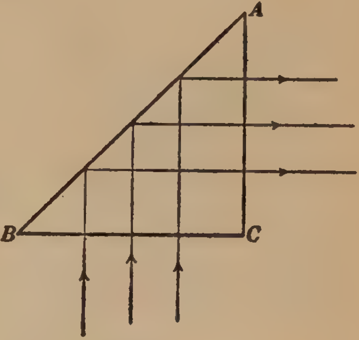

Fig. 42.

Total reflection takes place in Fig. 40, when the angle g is sufficiently large, for any two substances 1 and 2, if the velocity of light in substance No. 1 is less than the velocity of light in substance No. 2, that is to say, if G/A is less than unity, where G is the velocity of light in substance No. 1 and A is the velocity of light in substance No. 2, and the critical value of the angle g is that for which sin $g = G/A$. For example, if substance No. 1 is ordinary glass and if substance No. 2 is water, then $G/A = 8/9$ and total reflection takes place if the angle g is greater than that value for which sin $g = 8/9$.

Figure 42 shows what is called a total reflecting prism.

The brilliant silvery appearance of the surface of water in a glass tumbler when seen obliquely from below is due to total reflection.

43. Deflection of a ray of light by a glass prism. Figure 43 shows the path of a ray of light through a prism of glass. The total angle of deflection α depends on the index of refraction of the glass, on the angle ϕ of the prism, and on the direction of the ray in the prism. When the ray in the prism is equally inclined to the two faces of the prism, as shown in Fig. 43, the deflection α is a minimum, and in this case

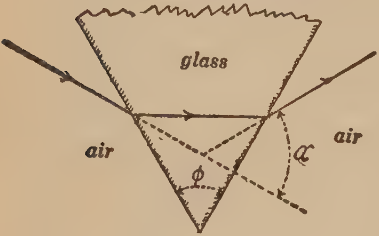

Fig. 43.

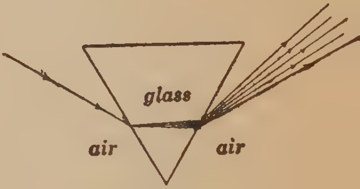

Fig. 44.

$$\mu = \frac{\sin \frac{1}{2}(\phi + \alpha)}{\sin \frac{1}{2}\phi} \quad \ldots\ldots\ldots\ldots\ldots \text{(i)*}$$

where μ is the index of refraction of the glass of which the prism is made. The most accurate method for determining the index of refraction of a sample of glass (for light of a given color or wave-length) is to make a prism from the sample, measure the angle ϕ of the prism and observe the angle α, whence μ can be calculated from equation (i).

Dispersion. As stated above, the angle α depends on the index of refraction of the glass, but the index of refraction of the glass has different values for different wave-lengths (colors) of light. Therefore if the incident ray in Fig. 43 contains light of different wave-lengths (or colors) each wave-length or color will be differently deflected by the prism as indicated in Fig. 44. This effect is called the *dispersion of light*, and it is discussed in Chapter VIII.

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* See Preston's *Theory of Light*, page 123.

44. Refraction of a spherical wave at a plane surface. This matter of the refraction of a spherical wave at a plane surface is much too complicated to be discussed in detail in this text, but some very interesting facts may be set forth by means of a careful drawing based on Huygens' construction. Such a drawing is reproduced in Fig. 45.

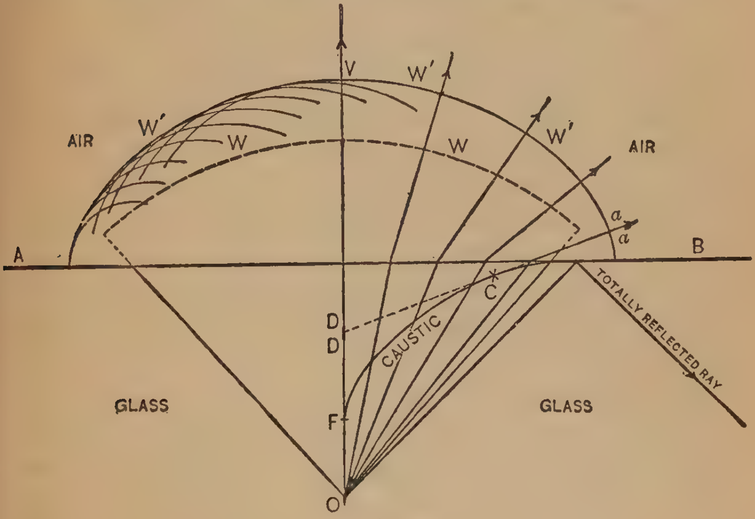

Fig. 45.

The point *O* in Fig. 45 represents a luminous point in glass (or in water), *WW* represents the position a spherical wave from *O* would have reached at a given instant if it had continued traveling in glass (or water), and *W'W'* shows the actual position of the wave in the air at the given instant. The wave $W'W'$ is not spherical, and a pencil of rays corresponding to the small portion *aa* of the wave front is an astigmatic pencil with one of its focal lines at C (perpendicular to the plane of the paper) on the caustic, and the other focal line at DD on the axis of symmetry of the figure. A small portion of the wave front near V is sensibly spherical, and the pencil of rays corresponding to a small portion of the wave front at V is a homocentric pencil with its focal point at F.

A very striking experiment is the following: A fine bit of chalk is placed at the bottom of a basin of water (at O, Fig. 45). With the eyes at *aa* the bit of chalk is seen at DD *if the line joining the eyes is horizontal*, and it is seen at C *if the line joining the eyes is vertical*. The familiar broken appearance of a straight oar in still, clear water is due to the fact that any point, like O, Fig. 45, of the submerged portion of the oar seems to be raised to DD.

45. Refraction at a spherical surface. A highly simplified and only approximately correct discussion of refraction at spherical surfaces is given in the chapter on lenses. A detailed discussion of this topic is beyond the scope of this text, but two cases are sufficiently simple and at the same time of sufficient interest to be given here, as follows.

Case I. The vertical dimensions of Fig. 46 are to be thought of as indefinitely small, and it is required to find the point P in glass at which a narrow pencil of rays from the point O in air will be focussed by the spherical surface SS whose center is at C and whose radius is r. Let A be the velocity of light in air and let G be the velocity of light in the glass. Then from equation (iii) of Art. 40

we have $\sin a/\sin g = A/G$, where a and g are the angles shown in Fig. 46. But these angles are very small so that $\sin a = a$ and $\sin g = g$. Therefore

$$\frac{a}{g} = \frac{A}{G} \quad \dots\dots\dots\dots\dots \text{(i)}$$

Let s represent the short distance from the axis to the point q where the ray from O strikes the spherical surface. Then the small angles o, p, and c are

$$o = \frac{s}{u} \quad \dots\dots\dots\dots\dots \text{(ii)}$$

$$p = \frac{s}{v} \quad \dots\dots\dots\dots\dots \text{(iii)}$$

and

$$c = \frac{s}{r} \quad \dots\dots\dots\dots\dots \text{(iv)}$$

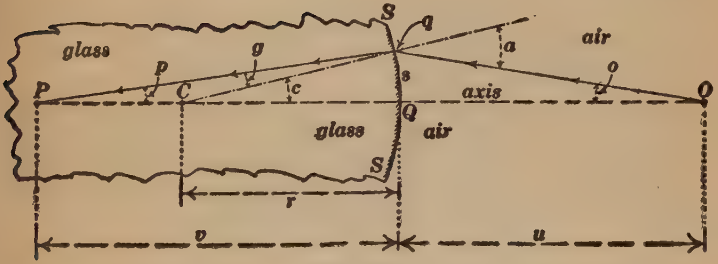

Fig. 46.

But, from simple geometrical considerations relating to the triangles PCq and OCq, we have

$$g = c - p \quad \dots\dots\dots\dots\dots \text{(v)}$$

and

$$a = c + o \quad \dots\dots\dots\dots\dots \text{(vi)}$$

Therefore, substituting the values of c and p and of c and o from (ii), (iii), and (iv) in (v) and (vi), and substituting the resulting values of g and a in equation (i), we get the desired relation between u, v, r, A and G.

Case II. There is one particular case in which a spherical wave remains accurately spherical after refraction at a spherical surface, and this case is of such importance in the high-power microscopic objective that it is worth while to set it forth without giving the complete proof.

Choose two fixed points F and F' in Fig. 47*a*, and consider any point p whose distance from F' is μ times its distance from F as indicated in the figure, where μ is any constant. Then it can be shown that the point p lies on a sphere, and it can be shown that the ratio OF'/OF is equal to μ^2.

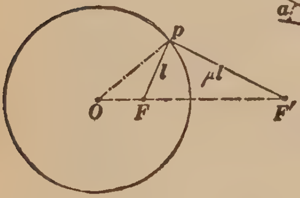

Fig. 47*a*.

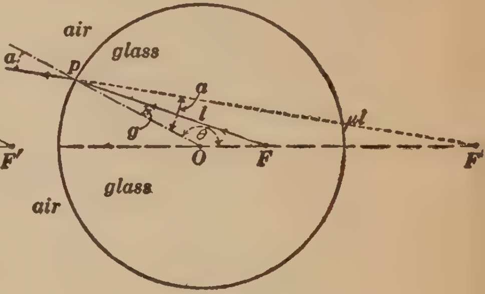

Fig. 47*b*.

Figure 47*b* shows a solid glass sphere surrounded by air, the index of refraction of the glass referred to air being μ. The full line arrow in Fig. 47*b* represents a ray of light emanating from F, and it is required to prove that after passing into the air it is directed away from F' as shown in the figure. From the triangle OpF we get

$$\frac{\sin g}{\sin \theta} = \frac{OF}{l} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

and from the triangle OpF' we get

$$\frac{\sin a}{\sin \theta} = \frac{OF'}{\mu l} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

whence, dividing (ii) by (i) member by member, we get equation (iii) of Art. 40, which shows that the rays from F are refracted so as to appear to have come from F', or, what amounts to the same thing, a spherical wave (in the glass) which has come from F becomes a spherical wave with its center at F' when it passes out of the glass into the air.

LEADING QUESTIONS

L. Q. 21. What is meant by diffuse reflection and refraction as contrasted with regular reflection and refraction ?

L. Q. 22. What is the great difference between Huygens' principle as applied in Art. 15 and Huygens' principle as applied in Art. 34 ?

L. Q. 23. What is meant, optically, by the image of an object in a mirror ?

L. Q. 24. What is Snell's law of refraction ?

L. Q. 25. State Fermat's principle as applied to refraction

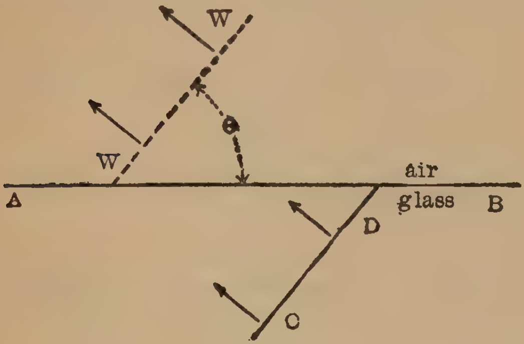

Fig. 48.

L. Q. 26. State Fermat's principle as applied to reflection.

L. Q. 27. Under what conditions does the phenomenon of total reflection occur ?

L. Q. 28. Figure 48 shows a plane wave *CD* in glass, and *WW* is the position this wave would have reached at a given instant if the glass were everywhere. Make a careful drawing of Huygens' construction for the actual position of the wave in air at the given instant (*a*) when the angle θ is 30°, and (*b*) when the angle θ is 60°. Take the index of refraction of the glass, *A*/G, equal to 1·50.

PROBLEMS

19. A 2-inch cube of glass *CC*, Fig. 49, is placed as shown in front of a scale *SS* so that the scale can be seen through the cube at *a*, and by looking over the cube at *b*; and the distance *ab*, as read off the scale, is 0·39 inch. Find the index of refraction of the glass of which the cube is made. Ans. 1·5.

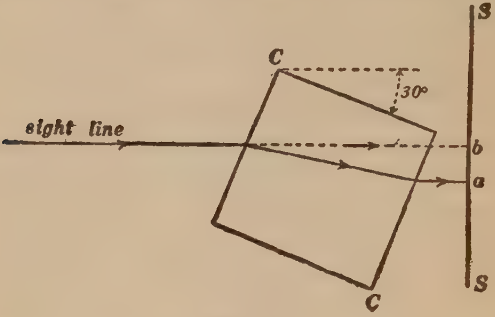

Fig. 49.

20. Find the least value of the ratio *CB*/*CA* in Fig. 50 for which the rays *r* (which are parallel to *CB*) will be totally reflected by the face *AB* of the prism: (*a*) When the prism is surrounded by air, and (*b*) When the prism is surrounded by a liquid whose index of refraction referred to air is 1·40, the index of refraction of the glass of the prism referred to air being 1·55.

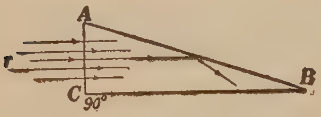

Fig. 50.

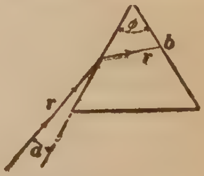

Fig. 51.

21. Find the largest value of the angle ϕ of the prism in Fig. 51 for which the ray *r* will pass through the prism (not to be totally reflected at *b*) when the angle *a* is very small, first, when the prism is surrounded by air and, second, when the prism is surrounded by water. The index of refraction of the glass referred to air is 1·50 and the index of refraction of water referred to air is 1·33.

22. A block *SS*, Fig. 52, of a substance of which the index of refraction is to be determined has a polished face and it is submerged in carbon bisulphide *LL* in a glass vessel *VV* which has a flat window *FF*. A sheet of paper *pp* partly surrounds the vessel so as to give a uniform illumination. Under these conditions a sharply defined line is seen in the field of the telescope *T* when the crystal is turned so that the critical angle of total reflection *t* is as shown. The block *SS* is supported at the end of a vertical axis which projects above the surface of the carbon bisulphide, passes through the center of a horizontal divided circle and carries a pointer which plays over the divided circle so that the angle turned to bring the block *SS* from the position shown in Fig. 52 to the position shown in Fig. 53 can be observed. This angle is thus found to be 125° and the index of refraction of the carbon bisulphide is 1·63. What is the index of refraction of the substance *SS* ? Ans. 1·445.

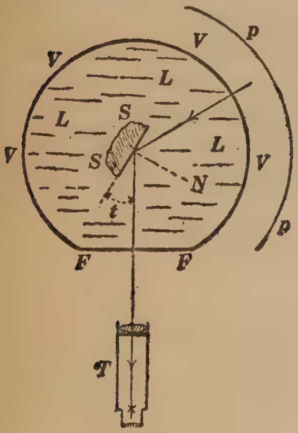

Fig. 52.

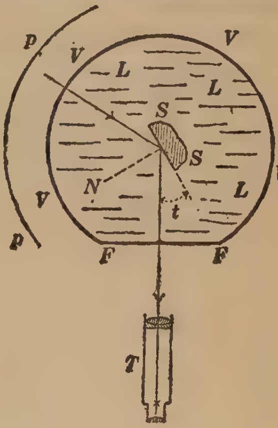

Fig. 53.

Note. The arrangement shown in Figs. 52 and 53 is extensively used for quick and easy measurements of indices of refraction. It is called a *refractometer*. A slightly modified form of the instrument can be used for measuring the index of refraction of a drop of liquid clinging to a glass surface.

23. Prove that the distance of the point F beneath AB in Fig. 45 is equal to $1/\mu$ times the distance of O beneath AB, using the method Case I of Art. 45.

24. Find the distance v in Fig. 46, when $u = 20$ centimeters, when the radius of the spherical surface SS is 10 centimeters, and when the index of refraction (A/G) of the glass is 1·50.

25. Find the distance v in the previous problem when u is infinity.

26. Rays parallel to PO in Fig. 46 come from the left. Find the distance from Q to the point where these rays are focussed in the air, the index of refraction (A/G) of the glass being 1·50.

CHAPTER V

SIMPLE LENSES

46. A lens is a portion of glass, or other transparent substance, bounded by plane or spherical surfaces* as shown in Figs. 54, 55, 56, 57, and 58. In these figures R represents the radius of curvature of the right face of the lens and L represents the radius of curvature of the left face of the lens. The *axis* of a lens is the line joining the centers of curvature of the surfaces of the lens as shown in Figs. 56, 75, and 58, or, it is the line passing through the center of curvature of one face of the lens and cutting the plane face of the lens at right angles as shown in Figs. 54 and 55.

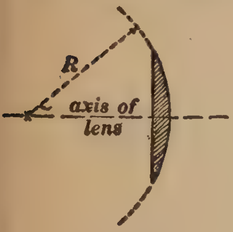

Fig. 54.

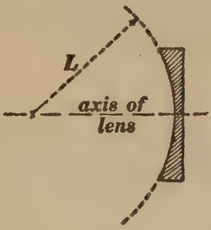

Fig. 55.

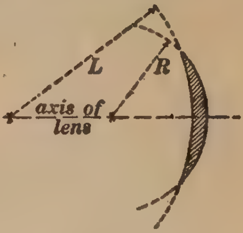

Fig. 56.

In the manufacture of lenses the rough lens is generally moulded from the desired quality of glass and then annealed. Each face of the lens is then accurately shaped by grinding the lens and a matrix together with every possible variety of sliding motion, thus bringing the surface of the lens automatically to almost perfect spherical shape. The polishing is accomplished by using finer and finer grinding material, usually powdered emery, and ending with rouge. The matrix is usually of iron, and in the later stages of the grinding and polishing the matrix is lined with a layer of stiff pitch with cross grooves cut in its surface.*

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* In some cases the surfaces of a lens are other than spherical in shape, but to make a surface conform accurately to any shape other than plane or spherical is a very tedious and difficult process.

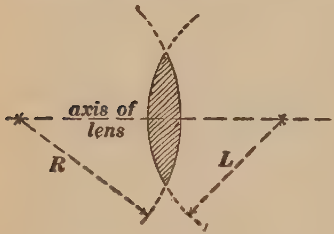

Fig. 57.

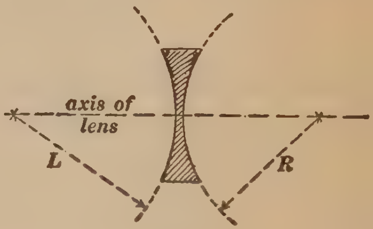

Fig. 58.

Converging lens. Figure 59 shows a train of plane light waves *AA* passing through a lens which is thick at the center and thin at the edge. The waves travel slower in glass than in air, so that the portions of the waves which

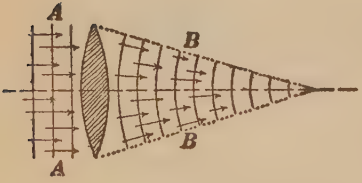

Fig. 59.

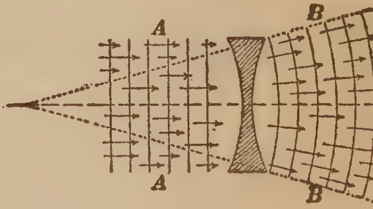

Fig. 60.

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* For a simple account of the processes of lens grinding see Lockyer's *Stargazing*, pages 117–138; see also "How to make a Refracting Telescope" in the *Scientific American Supplement*, Nos. 581, 582 and 583 (February 19 and 26 and March 5, 1887), pages 9283–9285, 9296–9299 and 9312–9316. A very interesting account of the manufacture of a large reflecting telescope is given by Henry Draper in the *Smithsonian Contributions to Knowledge*, No. 180, Vol. XIV, 1865.

pass through the thick part of the lens fall behind the portions which pass through the thin part of the lens; therefore the waves *BB* have *concave* fronts as shown, and they are concentrated at a certain point as indicated in the figure. A lens which is thick at the center and thin at the edge is called a *converging lens.*

Diverging lens. Figure 60 shows a train of plane light waves *AA* passing through a lens which is thin at the center and thick at the edge. In this case the central portions of the waves get ahead of the edge portions as they pass through the lens; therefore the waves *BB*, Fig. 60, have convex fronts as shown, and these waves appear to have come from a certain point as indicated by the dotted lines in the figure. A lens which is thicker at the edge than at the center is called a *diverging lens.*

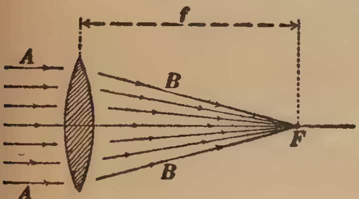

Fig. 61.

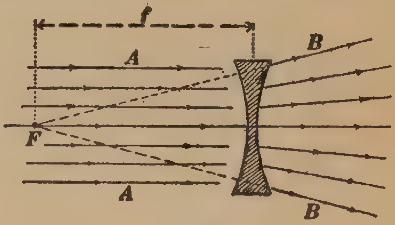

Fig. 62.

It is less confusing in lens diagrams to show the rays without showing the wave fronts. Thus Figs. 61 and 62 are duplicates of Figs. 59 and 60 respectively, but showing rays only.

47. Focal lengths. Focal points. Focal planes. The point *F* in Fig. 61 at which the beam of parallel rays *AA* is focussed by the lens (or the point *F* in Fig. 62 from which the beam of parallel rays *AA* appears to have come after passing through the lens) is called the *focal point* of the lens, the distance of *F* from the lens is called the *focal* *length* of the lens, and a plane through F at right angles to the axis of the lens is called the *focal plane* of the lens. A lens has, of course, two focal points (and two focal planes), one on each side of the lens.

In all lens equations the focal length of a converging lens is considered as a positive quantity, and the focal length of a diverging lens is considered as a negative quantity.

Definition of the diopter. Optometrists always specify the "power" of a lens in *diopters*, and the "power" of a lens so expressed is equal to the reciprocal of the focal length of the lens in meters. Thus a converging lens 2 meters in focal length has a "power" of + 0·5 diopter; a diverging lens 2 meters in focal length has a "power" of − 0·5 diopter.

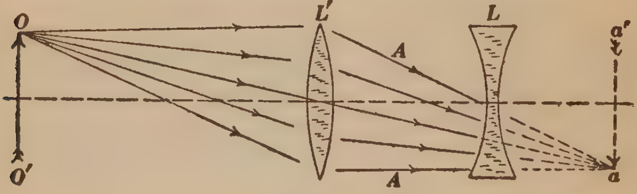

Fig. 63.

48. Limitations of this chapter. The ideal simple lens. Everyone is familiar with ordinary moving pictures in which images of a rapid succession of small transparent photographs are projected on a screen by a lens. This formation of an image by a lens is also exemplified in the photographic camera; an image of the object to be photographed is projected by the camera lens on a sensitive plate and this image is made permanent as a photograph. The projecting lens of a moving picture machine and photographic camera lenses of high grade are not *simple* *lenses*, they are compound lenses as explained in Chapter VII. *A simple lens produces a sharply defined image of an object only when the lens is very thin in comparison with its diameter,* when the light passes through the lens in a direction nearly parallel to the axis of the lens, and when light of one color (one wave-length) is used*, and the discussion of simple lenses in this chapter is subject to these limitations. In the diagrams, however, the lenses are shown as having considerable thickness, and the rays of light are frequently shown as being greatly inclined to the axis, all for the sake of clearness

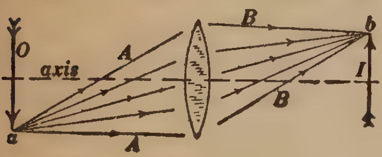

Fig. 64.

Real image of a real object. Light from *a* is focussed at *b*.

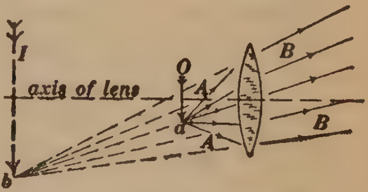

Fig. 65.

Virtual image of a real object. Light from *a* appears to have come from *b* after passing through the lens.

49. Image formation. The six possible cases of image formation by a lens are shown in Figs. 64 to 69, and the points *a* and *b* in each figure are called *conjugate points.*

In Fig. 64 both *a* and *b* are real.

In Fig. 65 the point *a* is real and *b* is virtual.

In Fig. 66 the point *a* is virtual and *b* is real.

In Fig. 67 the point *a* is real and *b* is virtual.

In Fig. 68 the point *a* is virtual and *b* is real.

In Fig. 69 both *a* and *b* are virtual.

---

* The lens is supposed to come to a sharp edge in Figs. 54, 56, and 57, or to be of zero thickness at the center in Figs. 55 and 58, for the sake of simplicity.

To understand Figs. 66, 68, and 69 one must know what is meant by a *virtual object*; thus *aa'* in Fig. 63 is an image of the object *oo'* which would be formed by the lens *L' if it were not for the lens L*, and *aa'* is called a *virtual object* with respect to the lens *L*. The rays *AA* travel towards the point *a* but never reach *a*, as shown in Figs. 63, 66, 67, and 69.

50. Geometry of image formation. The geometry of image formation by a simple lens can be developed from the facts set forth in Figs. 61–69 together with the following:

Proposition. *The straight line connecting a pair of conjugate points passes through the center of a thin lens.*

*Proof.** Consider the ray *R* in Fig. 70 which passes through the center of a thin lens. This ray cuts the surfaces of the lens at points where the surfaces are parallel to each other. Therefore the ray *R* passes through the lens exactly as if the lens were a flat glass plate as indicated in Fig. 71, and consequently the emergent ray *R'* is parallel to the incident ray *R*. But the sidewise displacement of the ray by the lens is negligible because the lens is supposed to be very thin. Therefore *R* and *R'* lie sensibly in one straight line. Now the ray *R* comes from the point *a* (or travels towards *a*, but never reaches *a*, if *a* is virtual), and the ray *R'* travels towards the point *b* (or appears to have come from *b*, if *b* is virtual). Therefore the points *a*, *C* and *b* lie on the straight line *RR'*.

Problem. *Given the position of the point a in any of the Figs.* 64 *to* 69 *and the focal length of the lens, to find by construction the position of the point b.*

The desired construction or drawing is based on three

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* This proof is given for the case of a lens whose surfaces have equal curvature, but the proposition is true for any very thin lens.

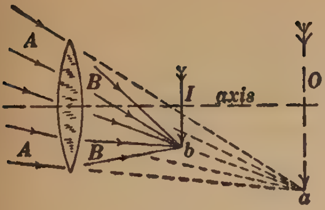

Fig. 66.

Real image of a virtual object. Light traveling towards *a* (but never reaching *a*) is focussed at *b*.

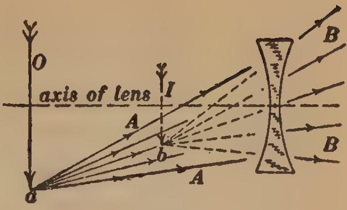

Fig. 67.

Virtual image of a real object. Light from *a* appears to have come from *b* after passing through the lens.

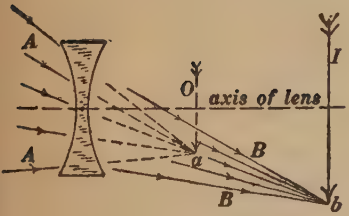

Fig. 68.

Real image of a virtual object. Light traveling towards *a* (but never reaching *a*) is focussed at *b*.

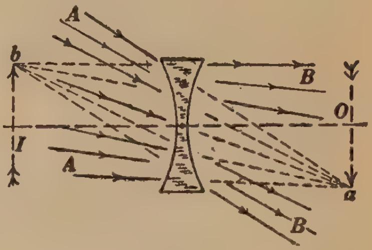

Fig. 69.

Virtual image of a virtual object. Light traveling towards *a* (but never reaching *a*) appears to have come from *b* after passing through the lens.

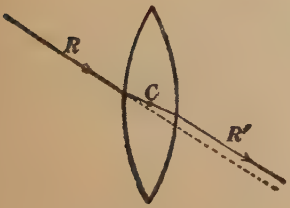

Fig. 70.

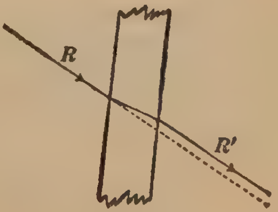

Fig. 71.

things, namely, (1) Any ray which passes through the center of a thin lens is not deflected by the lens, (2) any ray parallel to the axis of a converging lens passes through the right focal point *F* after passing through the lens as shown in Fig. 61, and (3) any ray parallel to the axis of a diverging lens appears to have come from the left focal point *F* after passing through the lens as shown in Fig. 62.

The actual construction or drawing is shown in Figs. 72 to 77. Select (from the bundle of rays *AA*) that particular ray 1 which passes through the center of the lens and extend this ray as ray 1′. Select (from the bundle of rays *AA*) that particular ray 2 which is parallel to the axis of the lens, and draw ray 2′ passing through the right or left focal point according as the lens is converging or diverging.

Relative dimensions of an object and its image. The ray from the object point *a* which passes through the center of the lens is undeflected by the lens, and this ray passes through the image point *b*. Likewise the ray from the object point *a*′ which passes through the center of the lens is undeflected by the lens, and this ray passes through the image point *b*′ as shown in Figs. 78 to 83. Therefore the diameter *aa*′ of the object is to the diameter *bb*′ of the image as the distance *u* is to the distance *v* (see Figs. 78 to 83).

51. Proposition. Let f be the focal length of a lens, let O and I be the semi-diameters of object and image as shown in Fig. 84, and let u and v be the distances shown in the figure, where a and b are conjugate points. Then

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v} \quad \text{or} \quad f = \frac{uv}{u+v} \ldots\ldots\ldots\ldots \text{(i)}$$

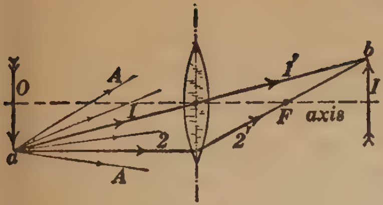

Fig. 72.
Corresponding to Fig. 64.

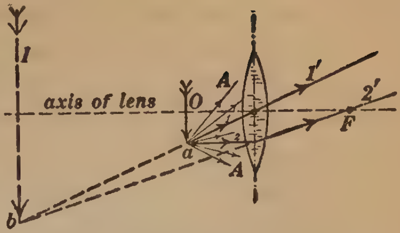

Fig. 73.
Corresponding to Fig. 65.

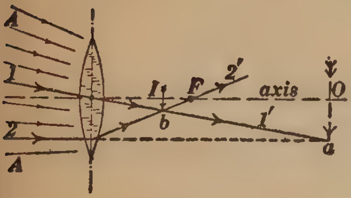

Fig. 74.
Corresponding to Fig. 66.

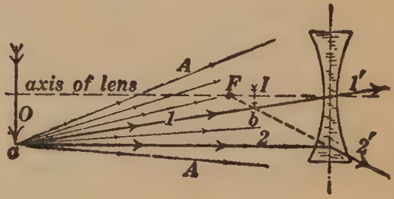

Fig. 75.
Corresponding to Fig. 67.

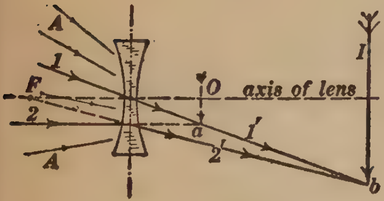

Fig. 76.
Corresponding to Fig. 68.

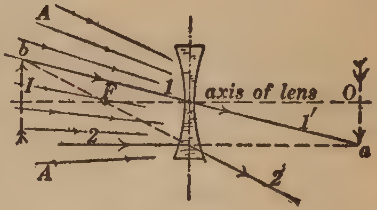

Fig. 77.
Corresponding to Fig. 69.

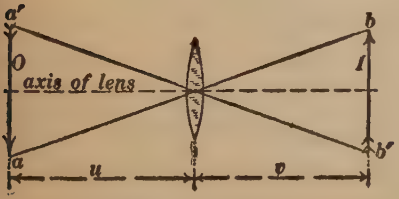

Fig. 78.
Corresponding to Fig. 64.

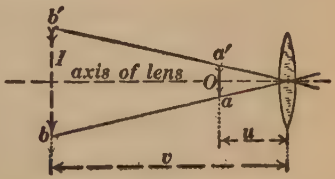

Fig. 79.
Corresponding to Fig. 65.

Proof. From Fig. 84 we have

$$\frac{O}{I} = \frac{u}{v} \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

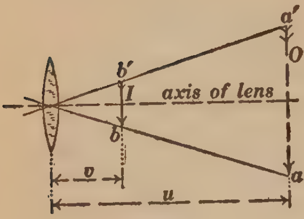

Fig. 80.
Corresponding to Fig. 66.

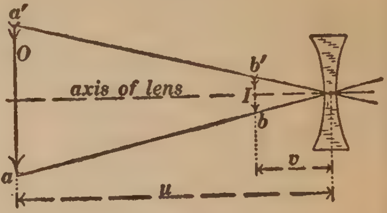

Fig. 81.
Corresponding to Fig. 67.

and, considering that the distance XC is equal to O, the two similar shaded triangles give

$$\frac{O}{I} = \frac{f}{v - f} \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

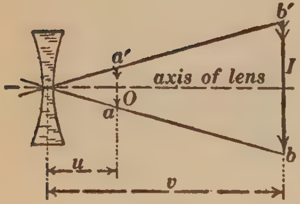

Fig. 82.
Corresponding to Fig. 68.

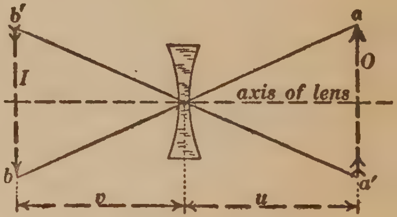

Fig. 83.
Corresponding to Fig. 69.

Therefore from equations (ii) and (iii) we have

$$\frac{u}{v} = \frac{v - f}{f}$$

which is equation (i) somewhat disguised.

Note. Equation (i) applies to converging and diverging lenses and to real and virtual points if the following conventions as to algebraic signs are adhered to :—

1. The focal length of a diverging lens comes out negative when calculated from equation (i), and the known value of the focal length of a diverging lens must be inserted in equation (i) with the negative sign.

2. The distance u (or v) from the lens to a virtual point comes out negative when calculated from equation (i), and the known value of u (or v) must be inserted in equation (i) with the negative sign if it refers to a virtual point.

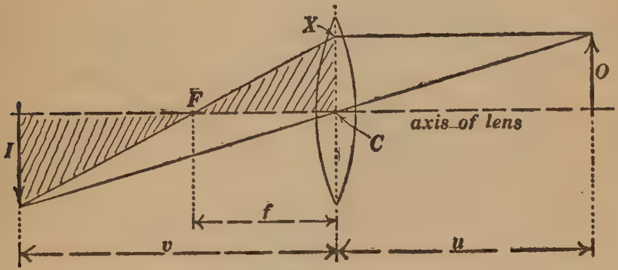

Fig. 84.

52. Proposition. The focal length, f, of a lens is given by the equation

$$f = \frac{d^2}{8(\mu - 1)h} \quad \ldots\ldots\ldots\ldots\text{(i)}$$

where d is the diameter of the lens measured to the sharp edge of the lens, h is the thickness of the lens at its center, and μ is the index of refraction, A/G (see Art. 33) of the glass.

Proof. A plane wave AA, Fig. 85, passes through the lens LL and is focussed at F. While the central part of the wave is passing through the central part of the lens (thickness h of glass) the portions of the wave which pass through the extreme edge of the lens will travel μ times as far or μh through air. Therefore **the effect of the lens is to cause the central portion of the wave to fall behind**

the edge portions *ee* by the amount $\mu h - h$ or $(\mu - 1)h$, as indicated in the figure.

The transmitted wave is sensibly spherical, and its radius of curvature f is the required focal length of the lens

From the triangle *Fec* we have

$$f^2 = \left(\frac{d}{2}\right)^2 + \Big[f - (\mu - 1)h\Big]^2$$

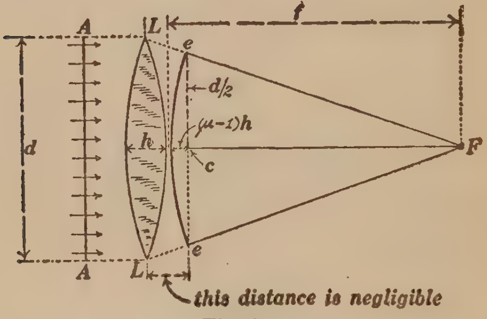

Fig. 85.

or, expanding and dropping the term containing h^2, we get

$$f^2 = \frac{d^2}{4} + f^2 - 2f(\mu - 1)h$$

from which equation (i) follows directly.

53. Proposition. The focal length, f, of a lens is given by the equation

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \ldots\ldots\ldots \text{(i)}$$

This equation may be derived from equation (i) of Art. 52 as follows: Figure 86 represents a lens which is understood to be very thin in comparison with its diameter d, but it is shown as a thick lens for the sake of clearness. From the shaded triangle we have

$$R_1^2 = \left(\frac{d}{2}\right)^2 + (R_1 - k_1)^2$$

or, expanding $(R_1 - k_1)^2$ and dropping the term which contains $k_1{}^2$, we have

$$R_1{}^2 = \left(\frac{d}{2}\right)^2 + R_1{}^2 - 2R_1k_1$$

from which we get

$$k_1 = \frac{d^2}{8R_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Similarly we may get

$$k^2 = \frac{d^2}{8R_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (iii)$$

where R_2 is the radius of curvature of the left-hand face of the lens in Fig. 86.

Then, substituting $h = k_1 + k_2 = \frac{d^2}{8R_1} + \frac{d^2}{8R_2}$ for h in equation (i) of Art. 52, we get equation (i) above.

Note. Equation (i) of this article is applicable to both converging and diverging lenses, but f is considered as negative for a diverging lens and the radius of curvature of a concave surface is considered as negative.

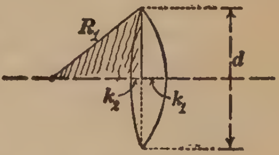

Fig. 86.

54. Effect of a lens on the candle-power of a beam of light. The most important practical case in which a lens (or its equivalent concave mirror) is used to modify the candle-power of a beam of light is the case of the ordinary headlight and a discussion of the headlight involves a careful consideration of the size of the light source. Therefore a simple ideal case will be discussed, first, in which the source may be considered to be negligibly small.

Light from an ideal point source A, Fig. 87, passes through a lens and is focussed at the point B. Beyond the point B the rays spread out in a cone gg whose opening (or spherical angle) is exactly the same as the opening (or spherical angle) of the cone ff. Let S be the area of the lens; then, approximately,* S/a^2 is the spherical angle of the cone ee in ster-radians, and S/b^2 is the spherical angle in ster-radians of the cone ff or of the cone gg. Therefore, if C is the candle-power (lumens per ster-radian) of the beam ee and if C' is the candle-power of the beam ff or of the beam gg, then the number of lumens of light flux in the cone ee is CS/a^2 and the number of lumens of light flux in the cone ff or in the cone gg is $C'S/b^2$. So that if no light is lost in the lens, we must have $CS/a^2 = C'S/b^2$, or

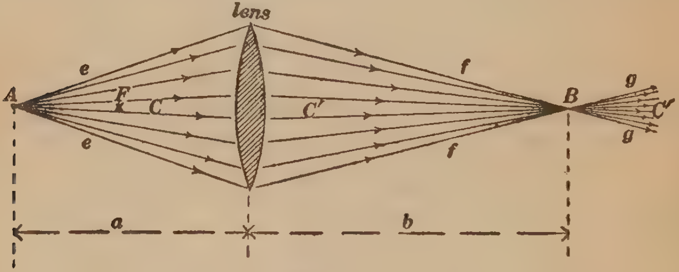

Fig. 87.

$$\frac{C}{a^2} = \frac{C'}{b^2} \quad \text{.....................(i)}$$

Remark. If a in Fig. 87 is equal to the focal length of the lens then the beam ff is a parallel beam, its spherical angle in zero, and its candle-power would be infinite if it contained more than an infinitesimal amount of light flux. *When a light source is at the focal point of a lens or mirror the size of the source cannot be neglected.*

---

* Because the area of the lens is very nearly equal to the area of the portion (inside of the cone ee) of the spherical surface of radius a, whose center is at A.

55. The searchlight. The searchlight (and also the ordinary headlight) consists of a lamp placed at the focal point of a concave mirror. The action of the concave mirror is, however, essentially the same* as the action of a converging lens, and, since the action of a converging lens is easier to show in a diagram than the action of a concave mirror, the following discussion is made to refer to a searchlight in which a converging lens is used. Also the lamp is taken to be an acetylene flame because of its simple shape.

Figure 88 shows an acetylene flame placed at the focal point (in the focal plane) of a converging lens. Then, neglecting lens imperfections, the light from each point *p* of the flame is converted into a beam of parallel rays as shown. Therefore the light from each point *p* illuminates a circular spot *ss* on a distant screen, and the diameter of this spot is equal to the diameter of the lens. The illuminated field on the distant screen is a blurred image of the lamp flame, the central portion *UU* of the field is uniformly illuminated (if the flame is uniformly

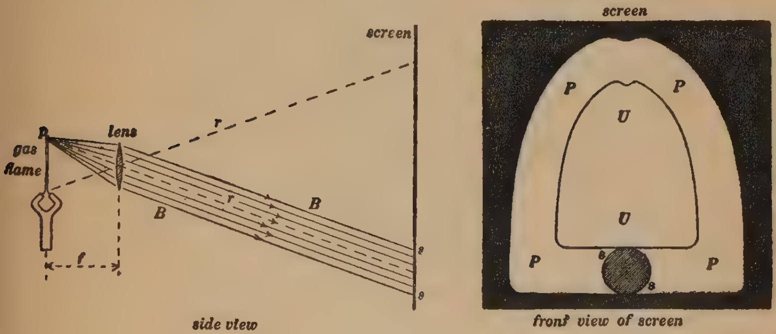

Fig. 88.

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* Except for aberration errors which are neglected in this discussion.

bright), and the fringe PP shades off gradually to complete darkness as shown in the figure.

The parallel beams like BB are very narrow as compared with the width of the distant illuminated field, and *therefore each parallel beam like BB can be thought of as a single ray.* From this point of view the searchlight beam consists of a divergent beam of rays which lie inside of the cone which is represented by the dotted lines rr in Fig. 88, and therefore, if the field UU is uniformly illuminated (if the lamp flame is uniformly bright), the searchlight beam has a definite candle-power C'. In fact, we have

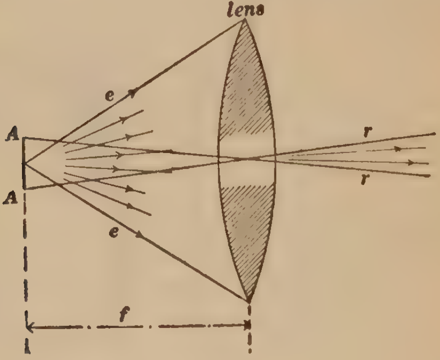

Fig. 89.

$$C' = CS/s \quad \ldots\ldots\ldots\ldots\ldots(\text{i})$$

where C' is the candle-power of the searchlight beam, C is the candle-power of the light from the bare lamp, S is the area of the searchlight lens (or mirror), and s is the area of the luminous surface of the lamp.

Proof of equation (i). Let AA, Fig. 89, be the lamp flame, and let C be the candle-power (lumens per steradian) of the beam ee. Let s be the area of the flame AA, and let S be the area of the lens. Then the opening of the cone *ee* is approximately S/f^2 ster-radians, and the number of lumens of light flux in the cone *ee* is CS/f^2. The cone *rr* which contains the searchlight beam (see Figs. 88 and 89) has an opening of s/f^2 ster-radians, approximately, and, if C' is the candle-power of this beam, it is evident that $C's/f^2$ is the number of lumens of light flux in the searchlight beam. But, neglecting losses of light, the light flux in the beam *ee* is equal to the light flux in the searchlight beam. That is to say, $CS/f^2 = C's/f^2$, or $CS = C's$, which is equation (i).

Example. A powerful searchlight has an arc lamp whose luminous area *s* is 0·25 square inch and which gives a 10,000 candle-power beam ($= C$ in the above discussion). The area of the lens (or mirror) is 300 square inches. Then the approximate candle-power of the searchlight beam is 12,000,000.

Remark. In the above discussion of the searchlight the aberration errors of the lens (or mirror) are neglected, whereas these errors are very considerable, especially in the case of deep parabolic reflectors.* The most important effect of these aberration errors is as follows: As stated above, the central portion of the searchlight beam in Fig. 88 has a definite candle-power if the flame is uniformly bright; but how about the searchlight beam when a closely coiled tungsten filament lamp is used? Except for aberration errors the front view of the distant screen in Fig. 88 would be a slightly blurred image of the coiled filament and the distribution of the rays in the divergent headlight beam would be so irregular that one could hardly think of the beam as having any definite candle-power; but the aberration errors of the lens (or mirror) cause such an excessive blurring of the image of the coiled filament on the distant screen as to give a nearly uniform illumination on the screen and consequently a fairly definite candle-power.

---

* Oblique spherical aberration is extremely large in the case of a deep parabolic reflector, even for rays only slightly inclined to the axis of the reflector.

LEADING QUESTIONS

L. Q. 29. Why are ordinary lenses always made with accurately spherical surfaces when a slight departure from true spherical shape would in most cases give a better lens?

L. Q. 30. A point source of light is placed at one of the focal points of a converging lens. Make a diagram showing the rays after passing through the lens.

L. Q. 31. What does it mean to say that the focal length of a diverging lens is 20 cm.?

L. Q. 32. What does it mean to say that the power of a spectacle lens is — 0·75 diopter?

L. Q. 33. Explain what is meant by a virtual object in the discussion of image formation by a lens.

L. Q. 34. A converging lens, only, can form a real image of a real object. Explain the physical reason for this.

L. Q. 35. A diverging lens, only, can form a virtual image of a virtual object. Explain the physical reason for this.

L. Q. 36. Consider the spot of light produced on a distant screen by a headlight beam (the headlight consisting of a lamp at the focal point of a parabolic reflector, no diffusing window). What does this spot of light represent and why is it not uniformly illuminated?

L. Q. 37. Under what idealized conditions is it permissible to think of a searchlight beam as having a definite candle-power?

PROBLEMS

27. An object, represented by a vertical arrow 10 centimeters long, is at a distance of 50 centimeters from a converging lens of which the focal length is 15 centimeters. Make carefully a drawing, to scale, showing the position and size of the image. Is the image real or virtual?

28. The object in problem 27 is placed 10 centimeters from the lens. Make carefully a drawing, to scale, showing the position and size of the image. Is the image real or virtual?

29. An object, represented by a vertical arrow 10 centimeters long, is at a distance of 50 centimeters from a diverging lens of which the focal length is 15 centimeters. Make carefully a drawing, to scale, showing the position and size of the image. Is the image real or virtual?

30. Rays of light coming from the left converge towards an image *b*, but never reach the image because they pass through a converging lens of which the focal length is 15 centimeters. The image *b* is 50 centimeters to the right of the lens and it is represented by a vertical arrow 10 centimeters long. Make carefully a drawing, to scale, showing the position and size of the image of *b*. Is this image real or virtual?

31. Rays of light coming from the left converge towards an image *b* but never reach the image because they pass through a diverging lens of which the focal length is 15 centimeters. The image *b* is 50 centimeters to the right of the lens and it is represented by a vertical arrow 10 centimeters long. Make carefully a drawing, to scale, showing the position and size of the image of *b*. Is this image real or virtual?

32. The image *b* in the previous problem is 10 centimeters to the right of the lens. Make carefully a drawing, to scale, showing the position and size of the image of *b*. Is this image real or virtual?

33. Two simple converging lenses each having a focal length of 15 centimeters are placed co-axially at a distance of 10 centimeters apart, and an object represented by a vertical arrow 10 centimeters long is placed at a distance of 25 centimeters from the middle point of the two lenses. Take carefully a drawing, to scale, showing the position and size of the image produced by the joint action of both lenses.

Note. Construct image produced by first lens, and then construct the image of this image produced by the second lens.

34. Show that the combined "power" in diopters of two lenses placed very close together is equal to the sum of the "powers" of the individual lenses in diopters.

Note. This demonstration is, perhaps, most easily developed by the method used in Art. 52. The effect of the two lenses together is to cause the central part of an originally plane wave to fall behind the edge portions of the wave by the amount $l + l'$, where l and l' are the amounts by which the respective lenses cause the middle part of an originally plane wave to fall behind the edge portions.

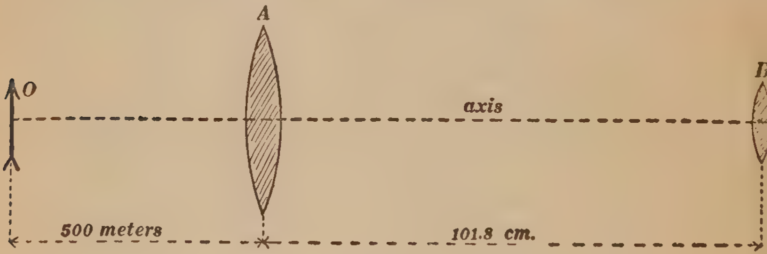

Fig. 90.

35. An object O, Fig. 90, is 10 meters in length. The focal length of lens A is 98·95 cm., and the focal length of lens B is 2 centimeters. Find the position and size of the image of O formed by the combination. Ans. The image is real and erect, it is 8 centimeters to the right of B, and its length is 5·95 cm.

Note. This problem shows how an ordinary simple astronomical telescope may be used to project a large image of the sun on a screen.

The dimension involved in this arrangement makes it very inconvenient to solve the problem by making a drawing. It is intended that this problem be solved by the use of equation (i) of Art. 51.

36. An object *O*, Fig. 91, is 10 meters in length. The focal length of lens *A* is 20 centimeters, and the focal length of lens *B* is 2 centimeters (negative). Find the position and size of the image of *O* formed by the combination. Ans. The image is real and inverted; it is 18·83 cm. to the right of *B*; and its length is 4·166 cm.

Note. It is intended that this problem be solved by using equation (i) of Art. 51. The arrangement in Fig. 91 is an ordinary opera-glass with its eyepiece pulled out slightly beyond its normal position, and its action is exactly like the action of the telephotographic lens (see Fig. 143). The next problem also refers to a converging lens and a diverging lens arranged as in Fig. 91.

37. A converging lens forms an image of a distant object at a place *A*, and when a diverging lens is placed 1 inch in front of *A*, a real image is formed 5 inches beyond *A*. Find the focal length of the diverging lens. Ans. – 1·25 inches.

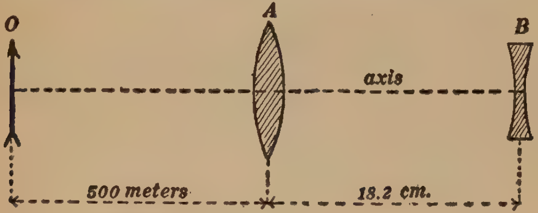

Fig. 91.

38. A lens is made of glass of which the index of refraction is 1·6. The diameter of the lens (measured to sharp edge) is 5 centimeters and the thickness of the lens at the center is 0·5 centimeter. What is its focal length?

39. A lens which comes to a sharp edge has a diameter of 10 centimeters and a thickness at center of 0·4 centimeter. The focal length of the lens is 62·5 centimeters. What is the index of refraction of the glass?

40. Figure 92 shows a beam *rr* of parallel rays focussed at the point *F* by the combined action of the two lenses *A* and *B*, and the single lens *C* of focal length *c* placed as shown would convert the beam *rr* into the same final convergent beam $r'r'$ as the two lenses A and B combined. Under these conditions the lens C is equivalent to A and B together, and the focal length of C is called the *equivalent focal length* of the combination AB. Show that $c = ab/(a + b - s)$, where a is the focal length of A, b is the focal length of B, s is the distance apart of lenses A and B, and c is the equivalent focal length of A and B together.

Note. The required derivation depends partly on equation (i) of Art. 51 as applied to lens B, with due regard for the fact that F' is a virtual point with respect to lens B.

The derivation depends also in part on two simple geometrical relations which may be seen in Fig. 92. One of these relations, namely, $u/a = v/c$, may not be seen at a glance, where c is the distance CF.

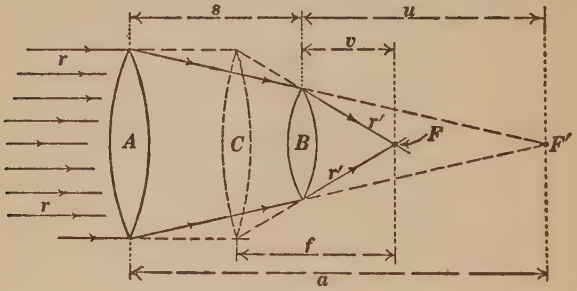

Fig. 92.

41. Show from equation (i) of Art. 53 that the ratio f/f' is the same for all colors (all values of μ) for two given lenses made of the same kind of glass, where f is the focal length of one lens for any particular color (wave length) and f' is the focal length of the other lens for the same color.

42. Using the result of the previous problem, show that the equivalent focal length of the combination AB in Fig 92 is the same for all colors if the lenses are made of the same kind of glass and if $s = \frac{1}{2}(a + b)$.

Note. See references to Huygens' eyepiece and to Ramsden's eyepiece at the end of Art. 77 and in connection with Figs. 135 and 136.

43. An approximately parallel beam of light having a sectional intensity of 20 lumens per square foot passes through a diverging lens whose focal length is -8 inches. Find the candle-power of the divergent beam produced, neglecting loss of light by reflection from the surfaces of the lens and by absorption in the material of the lens.

44. An open-arc electric lamp is placed 5 feet from a converging lens whose focal length is 16 inches, and the lamp beam has a conical intensity of 2500 candle-power. Treating the arc lamp as a point source and neglecting loss of light by reflection and absorption, find the candle-power of the beam which diverges beyond the image of the arc.

45. Two lamps A and B are placed at the ends of a Bunsen photometer bench, and the photometer screen is adjusted to give equal degrees of illumination on its two sides. The screen is then 100 cm. from lamp A and 300 cm. from lamp B. A converging lens whose focal length is 25 cm. is then placed 50 cm. from lamp B, and the screen and lamp B are left in their original positions. Find how far lamp A must now be placed from the screen to give equal intensities of illumination on the two sides of the screen. In solving this problem treat lamps A and B as point sources and neglect losses of light in the lens. Ans. 67 cm.

46. The lamp of a headlight gives a divergent beam of 30 candle-power, the lens (or parabolic mirror) is 8 inches in diameter, and the closely coiled tungsten filament of the lamp may be treated as a circular disk 0·25 inch in diameter. The headlight has no diffusing window. Find the approximate candle-power of the headlight beam, and find the distance of a vertical screen from the headlight in order that the central portion of the headlight beam may produce an intensity of illumination of 0·1 lumen per square foot on the screen.

CHAPTER VI

SIMPLE OPTICAL INSTRUMENTS

Note 1. The spectroscope, the polariscope, and the interferometer are important optical instruments which are described in subsequent chapters.

Note 2. In order to understand most easily the optical instruments which are described in this chapter the component lenses are to be thought of as simple lenses. As a matter of fact, however, the component lenses of optical instruments are always lens combinations (compound lenses) because simple lenses are very imperfect in their action, as explained in Chapters VII and VIII.

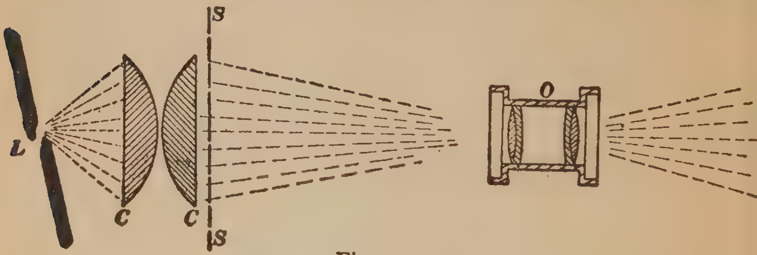

Fig. 93.

56. The photographic camera is an arrangement in which a lens projects a real image of an object on a sensitive photographic plate or film in a light-tight box. Photographic lenses are always compound lenses. See Chapters VII and VIII.

57. The projection lantern. The essential optical parts of a projection lantern are shown in Fig. 93. Light from a powerful lamp L passes through the two lenses CC, through a transparent picture SS, and through a lens O which projects an image of SS upon a wall or screen The two lenses CC (serving as a single lens) constitute what is called the *condensing lens*, the transparent picture SS is called a *lantern slide*, and the lens O is called the *object lens*. The function of the condensing lens is to direct the greatest possible amount of light through the transparent picture SS and through the object lens O, as shown by the dotted lines in the figure.

58. The eye.* Figure 94 shows a horizontal section of the human eye. The tough outer coating of the eye-ball is sharply curved and transparent in front forming the *cornea* NN. Between the cornea and the *crystalline lens* A is a clear watery fluid, the *aqueous humor* B. Behind the crystalline lens and filling the remainder of the eye-ball is a clear semi-fluid substance, the *vitreous humor* C. The front surface of the cornea and the two surfaces of the crystalline lens are sensibly spherical and they constitute a compound lens which projects an image of external objects

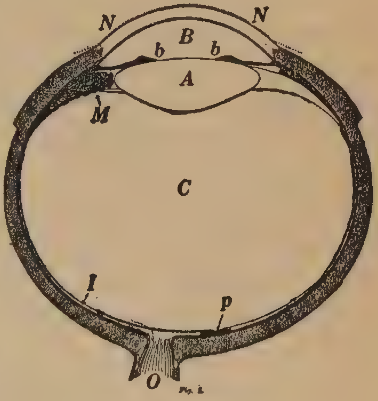

Fig. 94.

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* A good discussion of the structure of the eye and its functions; of the ophthalmoscope; of binocular vision and the stereoscope; and of the persistence of vision and the stroboscope is given in Edser's *Light for Students*, pages 159–196, Macmillan and Company. See also Müller-Pouillet's *Lehrbuch der Physik*, Vol. II, Part I (on light by Otto Lummer), pages 580–643. The great work on this subject is Helmholtz's *Handbuch der physiologischen Optik*, Leipzig, 1896.

upon a sensitive membrane, the *retina* I. The retina consists of a great number of minute end-organs of nerve fibers which enter the eye in a bundle at O, constituting what is called the *optic nerve.* The most sensitive part of the retina is a slightly depressed place at p which is oval in shape and about 0·5 mm. wide by 0·9 mm. long. It is called from its color the *yellow spot.* In the yellow spot the terminal organs of the optic nerve are packed very close together, and in order to see an object distinctly its image must fall upon this spot. Light which enters the eye passes through an aperture bb in a muscular membrane which is called the *iris.* The aperture bb is called the *pupil* of the eye.

Accommodation. In the photographic camera the distance of the lens from the sensitive plate is adjustable so that distant objects or near objects may be sharply focussed upon the sensitive plate at will. In the eye, however, the distance from the crystalline lens to the retina is invariable, and the lens of the eye is focussed by the action of a muscle M which surrounds the crystalline lens like a barrel hoop. The lens is, in youth, a bag of semi-fluid jelly, and when looking at a near object the muscle M squeezes the lens and makes it thicker at the center. This adjustment of the focus of the lens of the eye to give distinct vision is called *accommodation.* Ordinarily the eye has power of accommodation for objects at any distance greater than 15 centimeters from the eye. The distance of most distinct vision for the normal eye is about 25 centimeters. The accommodation of the eye is sensibly invariable for all distances exceeding eighty or ninety meters.

Imperfections of the eye. Some persons can accommodate the eye to distant objects with great effort or not at all; some persons can accommodate the eye to near objects with great difficulty, or not at all; some persons can accommodate the eye so as to see vertical lines or horizontal lines sharply but not so as to see both vertical and horizontal lines distinctly at the same time. A person who cannot accommodate his eyes to distant objects but who can accommodate his eyes to near objects is said to be *near sighted.* A person who can accommodate his eyes to distant objects but who cannot accommodate his eyes to near objects is said to be *far sighted.* Near-sightedness is relieved by the use of spectacles with diverging lenses, far-sightedness is relieved by the use of spectacles with converging lenses. Inability to see vertical lines and horizontal lines simultaneously is due to inaccurate "centering"* of the eye lenses or to more or less deviation from true spherical shape of the various refracting surfaces of the eye, and it is called *astigmatism.* Astigmatism is corrected by the use of spectacles having cylindrical surfaces.

59. Apparent size of an object. Visual angle. The highly sensitive part of the retina, the yellow spot *p* in Fig. 94, is very small, and the only portion of an object which can be seen distinctly is the portion whose image falls on this yellow spot as stated in Art. 58. In our discussion of the apparent size of objects, however, we will speak as if the whole retina were highly sensitive.†

An object appears large when its image covers a large portion of the retina, and the size of the image on the retina is determined by lines drawn from the extremities of the object through the center* of the lens of the eye as shown in Fig. 95. Therefore the angle α between these lines determines (or is a measure of) the apparent size of the object. This angle is called the *visual angle* of the object.

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* A compound lens system is said to be *centred* when the centers of curvature of all the spherical surfaces lie on the same straight line.

† The whole retina *is* highly sensitive to light but not highly sensitive in the perception of fineness of detail. Look sharply at the first letter in a word on a printed page and then look sharply at the last letter in the word! Note that you shift your eye because the yellow spot is so small that the image of only two or three letters fall upon it at once

60. The simple microscope or magnifying glass. Definition of magnifying power. The *simple microscope* or *magnifying-glass* is a converging lens which is held near the eye, and the object to be examined is moved until it is seen most distinctly. *The eye is then looking at an enlarged virtual image I of the object O as indicated in* Fig. 96.

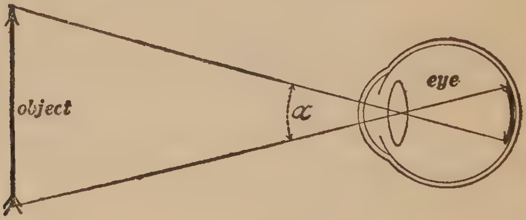

Fig. 95.

For the normal eye the distance of most distinct vision is about 25 centimeters, and *the standard magnifying power of a magnifying-glass is defined as the ratio I/O in Fig.* 96 **when the distance V is 25 centimeters.**

Proposition. The standard magnifying power, m, of a simple microscope is

$$m = \frac{25 - e}{f} + 1 \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

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* This statement is made as if the compound-lens effect of cornea and crystalline lens were equivalent to a single simple lens, which is not exactly true.

where f is the focal length of the magnifying-glass in centimeters, and e is the distance shown in Fig. 96. When e is negligible, equation (i) becomes

$$m = \frac{25 + 1}{f} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Proof of equation (i) The image I in Fig. 96 is virtual, and, therefore, from equation (i) of Art. 51, we have

$$\frac{1}{f} = \frac{1}{u} - \frac{1}{v} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

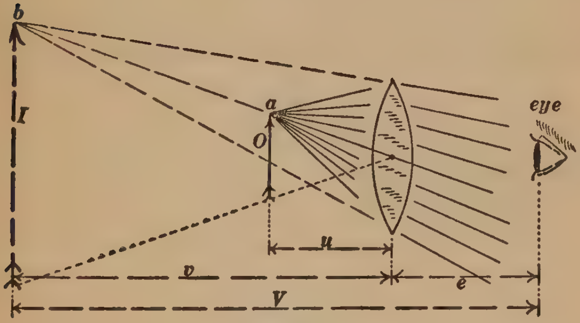

Fig. 96.

Also, from Art. 50, we have

$$m = \frac{I}{O} = \frac{v}{u} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

and, from Fig. 96, we have

$$v = V - e \quad \ldots\ldots\ldots\ldots\ldots \text{(v)}$$

Whence, using equations (iii) and (v) to eliminate u and v from equation (iv), we get equation (i).

Note. A high-power magnifying-glass must have a very short focal length according to equation (i), and it must be held very near to the object to be examined and very near to the eye. Magnifying-glasses are therefore inconvenient and unsatisfactory with magnifying powers exceeding about 25 or 30 diameters (focal lengths less than about 1 centimeter). Therefore for higher powers the compound microscope is used.

61. The compound microscope* *consists of a lens A, Fig. 97, which forms an enlarged real image I of an object O, and a magnifying-glass B for viewing this image.* The lens *A* is called the *object-glass* of the microscope, and the lens *B* is called the *eye-piece*. Both object-glass and eye-piece are shown as simple lenses in Fig. 97, but highly perfected compound lenses are always used for both *O* and *E*.

No attempt is made to show the actual paths of the rays in Fig. 97. The dotted lines are drawn so as to show the dependence of the ratio *I*/*O* upon the ratio *b*/*a*,

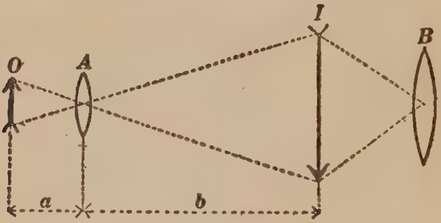

Fig. 97.

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* See *The Microscope and its Revelations*, by W. Carpenter (revised by W. H. Dallinger), London, 1901. An interesting outline of the historical development of the microscope is given in Hastings' *Light*, pages 83–110, Charles Scribner's Sons, 1901. Very good practical directions for the use and care of the microscope are published by the Bausch and Lomb Optical Company, of Rochester, New York. This publication is in the form of a small pamphlet, and it consists of extracts from a larger work, *The Manipulation of the Microscope*, by Edward Bausch.

A very widely used practical treatise on the use of the microscope is *The Microscope*, by Simon H. Gage, 13th edition, Comstock Publishing Co., Ithaca, N.Y., price $3·00 post paid.

A full discussion of the theory of the microscope by Czapski is given in Winkelmann's *Handbuck der theoretischen Physik*, Vol. VI, pages 328–373.

where I is the diameter of the image I and O is the diameter of the object.

The image I is looked at through the magnifying-glass (eye-piece) E exactly as if the image I were a real object like O in Fig. 96, and therefore the image I appears $\left(\frac{25}{f} \times 1\right)$ times as large as it would appear at a distance of 25 cm. from the naked eye, according to equation (ii) of Art. 60. But the image I in Fig. 97 is b/a times as large (in diameter) as the object O, and therefore the object O as seen through the compound microscope in Fig. 97 appears to be $(b/a) \times (25/f + 1)$ times as large (in diameter) as it would appear at a distance of 25 cm. from the naked eye.

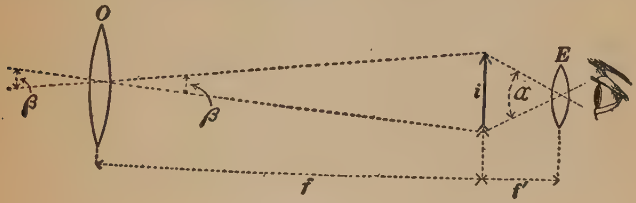

Fig. 98.

62. The telescope.* *The simple astronomical telescope consists of a long-focal-length converging lens O which forms an image i of a distant object* and a magnifying-glass E *for viewing this image,* as shown in Fig. 98. The lens O is called the *object-glass,* and the lens E is called the *eye-piece.* Both O and E are shown as simple lenses in Fig. 98, but both *O* and *E* are compound lenses in good telescopes.

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* A very interesting discussion of the telescope is given in Lockyer's *Stargazing,* pages 138–172, and in Hastings' *Light,* pages 53–82. See also *The Telescope,* by Louis Bell, McGraw-Hill Book Co., New York, 1922.

A full discussion of the theory of the telescope is given by Czapski in Winkelmann's *Handbuch der theoretischen Physik,* Vol. VI, pages 386–432.

No attempt is made to show the actual paths of the rays in Fig. 98; the dotted lines are drawn so as to show the visual angle β which the distant object would have if seen by the naked eye and the visual angle α of the distant object as seen through the telescope (see following discussion of Fig. 99).

The standard magnifying power of the telescope in Fig. 98 is the ratio α/β when the observer's eye is accommodated for parallel rays (distance *V* in Fig. 99 very great). Under these conditions the magnifying power $m = \alpha/\beta$ is given by the equation

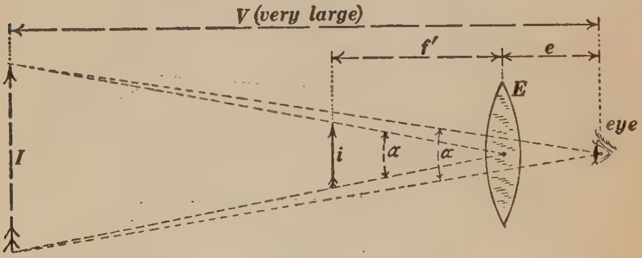

Fig. 99.

$$m = \frac{f}{f'} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

where *f* is the focal length of lens *O* in Fig. 98 and *f′* is the focal length of lens *E*.

Discussion of equation (i). Inasmuch as the standard magnifying power of the telescope of Fig. 98 is defined as the ratio of α/β when the observer's eye is accommodated for parallel rays it is necessary to discuss Fig. 96 for the case in which *V* is infinity. Under these conditions two things in particular are to be noted, namely, (*a*) the distance *u* in Fig. 96 is equal to *f′* as shown in Fig. 99, and (*b*) the two angles α in Fig. 99 are equal.

Figure 99 is a reproduction of Fig. 96, except that the image *i* in Fig. 99 takes the place of an actual object *O* in Fig. 96, and *V* is supposed to be very great. Figure 99 shows that the angle subtended by I from the center of E is equal to the angle subtended by I from the center of the observer's eye, which means that the visual angle of the distant object as seen through the telescope of Fig. 98 is equal to the angle α in Fig. 98.

Now the angles β and α in Fig. 98 are both very small, and therefore $\beta = i/f$ and $\alpha = i/f'$, so that $\alpha/\beta = f/f'$.

63. The erecting telescope or spy-glass. The simple telescope which is described in Art. 62 shows objects inverted. The *spy-glass* is a telescope so modified as to make distant objects appear erect. Figure 100 shows the essential parts of a spy-glass. The object-glass O forms at i an inverted image of a distant object, the lens S forms at i' an inverted image i, or an erect image of the distant object, and this erect image i' is viewed by the magnifying glass E. The introduction of the erecting lens S lengthens the telescope very considerably, and the spy-glass is consequently very long. In practice each of the lenses O, S, and E, Fig. 100, is a compound lens.

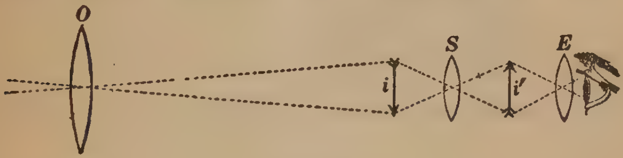

Fig. 100.

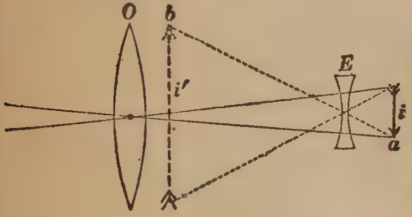

Fig. 101.

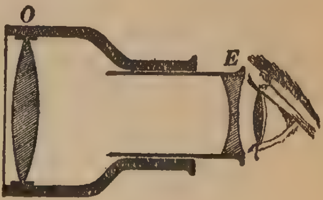

Fig. 102.

64. The opera glass is a telescope of which the eye-piece is a diverging lens, as shown in Figs. 101 and 102.

The object-glass O would form at i an inverted real image of the distant object if it were not for the interposition of the diverging lens E, the eye-piece. Light from a point in the object converges towards the point a after passing through the object-glass O, and the effect of the eye-piece is to cause the pencil of rays which is converging towards a to appear to have come from the point b, that is to say, the lens E forms an inverted enlarged virtual image at i' of the inverted virtual image i. The image i' is therefore erect, and the eye in looking through the lens E sees the distant object in an erect position. The action of the lens E in Fig. 102 is as shown in Figs. 69 and 77. The lens O is always a compound lens; but in the cheaper grades of opera-glasses the lens E is a simple lens.

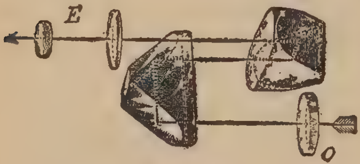

Fig. 103.

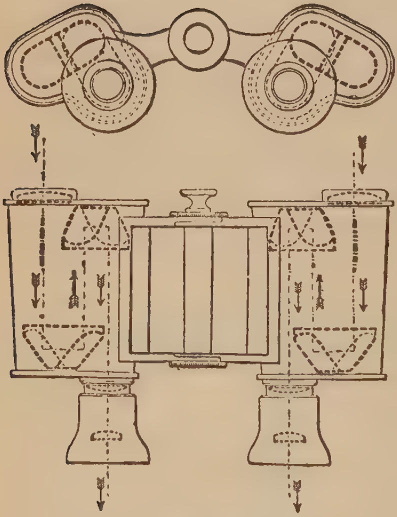

Fig. 104.

65. The telescope with Porro prisms. The great advantage of the opera-glass type of telescope is its shortness. An arrangement which was devised by Porro in 1852 makes it possible to greatly shorten the simple telescope of Fig. 98, and Porro's arrangement also makes a distant object appear erect. The essential feature of this arrangement is a pair of total reflecting prisms as shown in Fig. 103. A binocular telescope, or field-glass of this type, is shown in Fig. 104.

66. The use of the telescope for sighting. As used in the surveyor's transit and level, and as used in a great variety of astronomical and physical instruments, the telescope is used solely for accurate sighting. Very fine wires or spider lines are stretched across the focal plane *pp* of the object-glass as shown in Fig. 105. These cross-wires are in the same plane as the image of the distant object (as formed by the object-glass). Therefore the cross-wires and the image are both seen distinctly by the observer as he looks through the eye-piece, and the observer can move the telescope until the point where the two wires cross each other is accurately coincident with any desired point *a* in the image; when this is done *the line drawn from the crossing point of the wires through the center* of the object-glass passes through the corresponding point b of the distant object.*

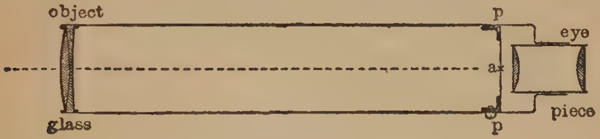

Fig. 105.

Sighting cannot be accurately done when the two "sights" consist of pins or point-like projections as on a gun, partly because it is impossible to focus the eye sharply on both sights and on the distant object simultaneously. The most accurate sighting that can be done by the naked eye in this way is with an error of about one minute of angle (about one inch at a distance of 100 yards). With a telescope the extreme accuracy of sighting is with an error of a few hundredths of a second of angle, less than a thousandth of an inch at a distance of a hundred yards.

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* The object-glass is here assumed to be an ideal simple lens.

LEADING QUESTIONS

L. Q. 38. Why is an object seen erect when its image on the retina is inverted? This question is as absurd as to ask "Why, when one hears a baby cry, with two ears, one does not think it is twins?" Our sensations in their usual forms are symbols which we have learned by experience to interpret in accordance with the facts.

L. Q. 39. A man a mile away looks as small as a one-inch doll at a distance of 100 feet. Why do we not *think* of the man as being only one inch tall? This is, of course, a question in psychology. See **L. Q. 38.**

L. Q. 40. What is meant by the visual angle of an object?

L. Q. 41. What is meant by the standard magnifying power of a microscope?

PROBLEMS

47. A photographic lens of 15 centimeters focal length gives a sharp image of very distant objects when the slider carrying the lens is in a marked position. Calculate the distance the slider must be moved so that an object two meters in front of the lens may be in sharp focus on the photographic plate. Ans. 1·2 centimeters forwards.

Note. In solving this and the following problems treat the lens as if it were a simple lens. The lens must be 15 centimeters in front of the photographic plate for very distant objects, and b centimeters in front of the plate for an object at distance a in front of the lens. The distance b is greater than 15 centimeters, and the difference $b - 15$ is the required distance that the lens must be moved.

48. A projection lantern takes a transparent slide or picture 7 × 7 centimeters, and the focal length of the lantern object lens is 20 centimeters. It is desired to project on the screen an image of the slide two meters square. Required the distance from screen to lens. Ans. 592 centimeters.

49. A projection lantern is to be used with its object lens 3 meters from a screen, and it is desired to project a 7 × 7 centimeter slide as a picture 2 meters square. What focal length object lens is required? Ans. 10·46 centimeters.

50. A far-sighted person can see distinctly an object at a distance of 200 centimeters. Find the power in diopters of a spectacle lens which will enable this person to see clearly an object at a distance of 25 centimeters from the eye. Ans. 3·5 diopters positive.

Note. The problem is to have a lens in front of the eye (distance from lens to eye being considered as negligibly small) so as to form a virtual image at 200 centimeters of an object at 25 centimeters from the eye. It is intended that this and the following problem be solved by equation (i) of Art. 51.

51. A near-sighted person can see an object distinctly when it is 15 centimeters from the eye. Find the power in diopters of a spectacle lens which will enable this person to see clearly an object which is 200 centimeters from the eye. Ans. 6·17 diopters negative.

Note. The problem is to have a lens in front of the eye which will form a virtual image at 15 centimeters from the eye of an object 200 centimeters from the eye.

52. A pocket magnifier has 3 separate lenses of which the focal lengths are 1, 2, and 4 centimeters respectively. What is the magnifying power of each when the eye is accommodated for a distance of 25 centimeters?

53. An object 0·1 millimeter in diameter, actual size, is looked at through a microscope of which the standard magnifying power is 30 diameters. What is the diameter of a drawing of the object which when placed at a distance of 3 meters from the naked eye will appear the same size as the object as seen through the microscope? Ans. 3·6 centimeters.

54. A compound microscope has an objective of which the focal length is 2 millimeters, an eye-piece of which the focal length is 10 millimeters, and the distance from center of objective to the plane of the image (*b* in Fig. 97 of Art. 61) is 150 millimeters. Calculate the magnifying power of the instrument on the assumption that the observer's eye is accommodated for a distance of 25 centimeters. Ans. 1924 diameters.

Note. The limit of effective magnifying power of a microscope is about 900 diameters. With a magnifying power of 1924 diameters the image would be perceptibly blurred even if the microscope itself were perfect. See Art. 80, section *d*.

55. If the distance *b* of Fig. 97, which is given as 150 millimeters in the previous problem, is increased to 20 centimeters, what would the magnifying power of the microscope be?

56. The object-glass of the great telescope of the Lick Observatory is 1500 centimeters in focal length. What is the magnifying power of the telescope when an eye-piece is used whose focal length is 2 centimeters. The moon's distance is 240,000 miles; at what distance would it have to be from the naked eye to appear to be the same size as when seen through the Lick telescope with a 2-cm. eye-piece?

57. A properly focussed telescope (image *i* at distance *f'* from *E* in Fig. 98, where *f'* is the focal length of *E*) is sighted at the sun. How far and in what direction must the eye-piece be moved so as to project a sharp image of the sun on a screen about 300 cm. back of the eye-piece, the focal length *f'* of the eye lens *E* being 2 cm.? Ans. 0·0132 cm. away from *O* in Fig. 98.

58. A sun spot has a diameter equal to 0·001 of the sun's diameter (0·0005°). Find the diameter of the image of this spot on the screen in the previous problem, the focal length of the object-glass *O* in Fig. 98 being as specified in problem 56. Ans. 1·96 cm.

59. An opera-glass is properly focussed (distance *Ei* in Fig. 101 equal to the focal length of *E*) and sighted at the sun. Find how far and in what direction the eye lens *E* must be moved to project a sharp image of the sun on a screen about 30 cm. back of the eye-piece. Take the focal length of *E* equal to 1·5 cm. (negative). Ans. 0·0714 cm. away from *O* in Fig. 101.

60. Find the diameter of the image of the sun on the screen in the previous problem, the angular diameter of the sun being ½ degree, and the focal length of the object-glass *O* in Fig. 101 being 10 cm. Ans. 1·83 cm.

61. The focal length of object-glass and eye-piece of a telescope are 30 cm. and 2 cm. respectively. An observer with his eye 5 cm. back of the eye-piece looks through the telescope at a millimeter scale *A*, which is 300 cm. in front of the object-glass of the telescope, and with his other eye the observer looks directly at a millimeter scale *B*, which is 55 cm. from his eye. Under these conditions both eyes will be similarly accommodated (both for a distance of 55 cm.), and the observer will see the two scales superposed. Find the length on scale *B* of one millimeter on scale *A*. Ans. 2·89 millimeters.

CHAPTER VII

LENS IMPERFECTIONS AND THEIR CORRECTION BY COMPOUNDING*

67. Focal imperfections of a perfect lens. In the discussion of the theory of lens imperfections it is important to understand that *light is focussed at a more or less* **blurred spot** *instead of a* **point** *because of the nature of light itself, and in spite of ideal perfection in the performance of a lens.*

A pebble is dropped at the center of a circular tub of water, a series of circular ripples pass out and are reflected by the wall of the tub, and *WW*, Fig. 106, represents a wave front on one of the reflected ripples as it converges towards the point *F*. The wave-energy of the reflected ripple is not, however, concentrated at the mathematical point *F*. A single ripple has some breadth and a series of ripples has a definite wave-length, and the wave-energy in Fig. 106 is concentrated at a *spot* the dimensions

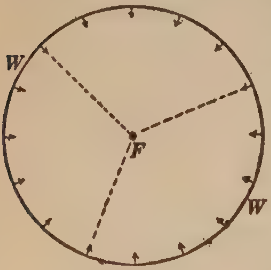

Fig. 106.

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* A good discussion of this subject is to be found in Lummer's *Photographic Optics*, translated by S. P. Thompson, Macmillan & Co., 1900. See also Drude's *Theory of Optics*, pages 31–92, translated by Mann and Millikan, Longmans, Green & Co., 1902. A very complete discussion of this subject by Czapski, Eppstein, and von Rohr is given in the second edition of Winkelmann's *Handbuch der Physik*, Vol. VI, on Optics, pages 1–470. See also Müller-Pouillet's *Lehrbuch der Physik*, ninth edition, Volume II, Part I (Optics), pages 443–880, by Otto Lummer.

of which are of the same order of magnitude as the breadth of a single ripple or the same order of magnitude as the wave-length of a series of ripples. So it is in the case of light waves. A complete and perfectly spherical wave as represented by *WW* in Fig. 106 would be focussed at a spot the dimensions of which are of the same order of magnitude as the wave-length of the light.

When the incoming wave front is not a complete sphere, as shown in Fig. 106, but a sector of a sphere as shown by the circular segments *WW* in Figs. 108 and 109, then the focal spot is greatly enlarged by diffraction.

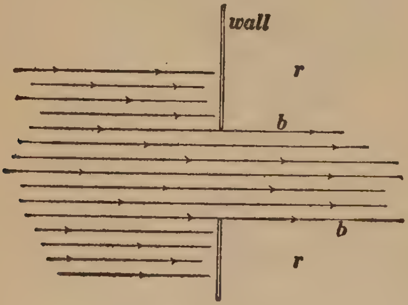

Fig. 107.

Diffraction. When light passes through a hole in a wall, as shown in Fig. 107, it is ordinarily supposed *not* to spread out into the regions *rr* but to go straight forwards as a beam with perfectly sharp boundaries *bb*. As a matter of fact, however, the light does spread out into the regions *rr*. Light bends around a corner in precisely the same way that water waves bend around a corner. This effect is called *diffraction*, but in the case of light, diffraction is on so small a scale that it is ordinarily inappreciable.

To get some idea of the effect of diffraction on the size of the focal spot of a perfect lens let us consider Figs. 108 and 109 which show a large diameter lens and a small diameter lens of the same focal length. *A very considerable portion of the wave energy leaks off,* as it were, from the edges of the waves WW into the surrounding rr.* **This is especially the case in Fig. 109 because the waves** ***WW*** **in Fig. 109 have a great deal of " edge " and but little " body."** As a matter of fact the focal spot *F* is much larger in Fig. 109 than it is in Fig. 108. This focal spot is a bright central spot surrounded by colored rings.

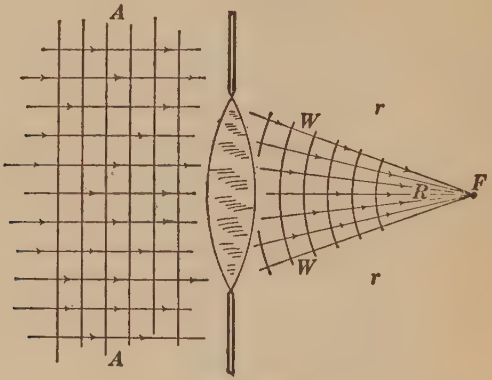

Fig. 108.

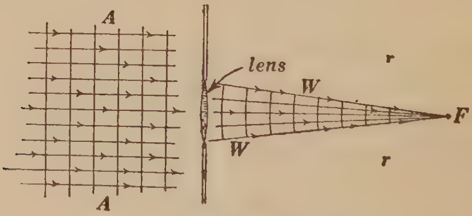

Fig. 109.

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* The terms used in this statement are necessarily very crude. The mathematical theory of diffraction is much too complicated to be given here. See Preston's *Theory of Light.*

*It is evident from the above discussion that focal imperfections due to the nature of light itself (due to diffraction) are reduced to the utmost by making the diameter of the lens large in comparison with its focal length, the lens action itself being supposed to be perfect.**

68. Numerical aperature. The free diameter† *d* divided by the focal length *f* of a lens is called the *numerical aperture* of the lens. See Fig. 110. Thus the object-glass of the great telescope of the Lick Observatory is 3 feet in diameter (= *d*) and its focal length is 50 feet (= *f*), so that its numerical aperture is 0·06. The best objectives for instantaneous photography have a numerical aperture of about 0·25, that is the free diameter is about one-quarter of the focal length. The numerical aperture of the best high-power microscope objectives is 1·4 or even 1·6.

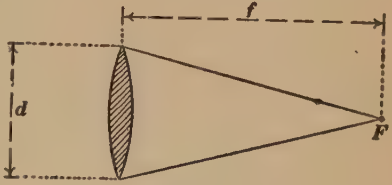

Fig. 110.

Large numerical aperture is described for two reasons, namely, (*a*) for the reason explained in Art. 67, when one wishes to get the greatest possible detail in an image formed by a highly perfected lens, and (*b*) to get a bright image, when a bright image is desired, as in instantaneous photography.

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* A very interesting and extremely clear discussion of this matter is given by Michelson, *Light Waves and Their Uses*, pages 27–30, University of Chicago Press, 1903.

† The free diameter of a simple thin lens is the diameter of the lens. The "free diameter" of a compound lens is not so easily defined. The definition of numerical aperture as here given is sufficient however for present purposes. For a full discussion of aperture see Drude's *Theory of Optics*, translated by Mann and Millikan, pages 73–92; see also Dallmeyer's *Telephotography*, pages 91–101. A very full discussion of this subject is given in Winkelmann's *Handbuch der Physik*, Vol. VI, pages 211–260, 298, and 347.

Under certain conditions, however, a small numerical aperture is necessary, namely, (*a*) when one wishes to get a fairly sharp image by means of a simple lens or by a poorly designed compound lens, or (*b*) when one wishes to produce on a photographic lens a fairly sharp image of a group of objects at different distances* from the camera or of a group of objects spread over a wide field. Every photographer knows that the very best of compound lenses do not give sharp images of such groups unless the free diameter of the lens is reduced by a diaphragm or "stop." See Art. 79.

The brightness of the image which a lens forms of a given distant object is proportional to the square of the numerical aperture of the lens. Consider two lenses *A* and *B* of the same focal length, the diameter of lens A being twice as great as the diameter of lens *B* (numerical aperture twice as great). These two lenses form images of the same size of a given distant object, and, inasmuch as the larger lens gathers four times as much light, its image is four times as bright. Consider two lenses *A* and *B* of the same diameter, lens *A* having a focal length twice as great as lens *B* (numerical aperture half as great). Under these conditions the lenses gather the same amount of light, but lens *A* forms an image twice as large in diameter or four times as large in area as that which is formed by lens *B*, so that the image formed by lens *A* is one-quarter as bright as the image formed by lens *B*.

69. Field angle. The angle between lines drawn from the center* of a lens to the extreme edges of the largest distinct image which the lens can produce is called the *field angle* of the lens. Thus the angle *F* is the field angle of the lens in Fig. 111. A simple lens usually gives an image so badly blurred that one can scarcely speak of any distinct image at all, and such a lens becomes extremely unsatisfactory for field angles greater than a few degrees. The field angle of high-grade telescope and microscope objectives is ordinarily very small, seldom exceeding one or two degrees. Some photographic objectives, on the other hand, are made which give excellent definition with a field angle as great as 135 degrees. Such lenses are called *wide angle lenses*.

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* This relates to what the photographer calls "depth of focus." Depth of focus is practically independent of refinements in the design of a compound lens, it depends only on numerical aperture and focal length.

It is impossible to eliminate certain imperfections of a lens which is used as a wide angle lens, and therefore such lenses are always used with small numerical aperture in order to make the image reasonably sharp and distinct.

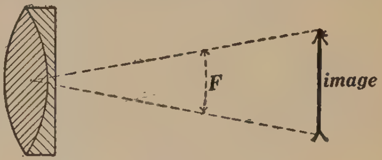

Fig. 111.

70. Spherical aberration. A plane or spherical wave is in general not plane or spherical after passing through a lens. This effect is called *spherical aberration*. There is, of course, an infinite variety of ways in which a wave front (or any surface) may be *not* spherical or plane, and spherical aberration is, in fact, an extremely complicated thing.

A very narrow pencil of rays passing through a lens parallel to the axis of the lens is not subject to spherical aberration because, in the first place, the transmitted waves are symmetrical about the axis (equal curvature in every direction), and, in the second place, the transmitted waves are so small that they cannot be distinguished from sectors of a spherical surface.

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* This statement is made as if we were dealing with a simple lens.

A *broad beam* of parallel rays *parallel to the axis of a lens* is subject to spherical aberration to a very considerable degree, as described in the next article.

A *narrow pencil* of rays *which passes obliquely through a lens* becomes a well-defined astigmatic pencil, as described in Art. 72.

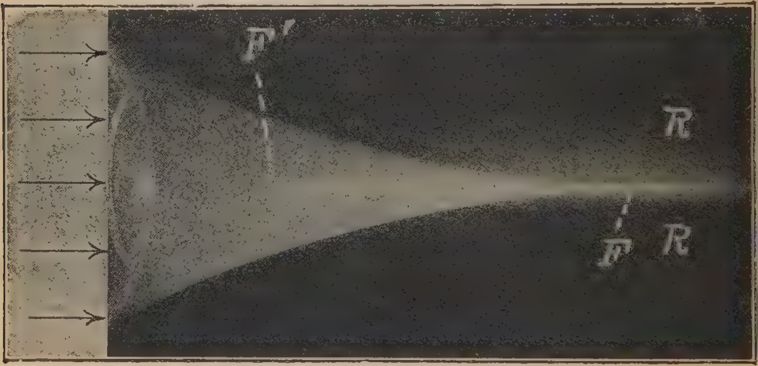

Fig. 112.

A *broad oblique pencil* or beam of rays is very greatly confused by a simple lens, as described in Art. 73.

There are, therefore, three kinds of spherical aberration, namely, (*a*) spherical aberration of a broad pencil or beam of rays parallel to the axis of the lens, (*b*) spherical aberration of a narrow oblique pencil of rays, and (*c*) spherical aberration of a broad oblique pencil or beam of rays. The first is called *axial spherical aberration*, the second is called *astigmatism*, and the third is called *oblique spherical aberration*, or *coma*.

71. Axial spherical aberration. Figure 112 is a photograph of a cloud of white smoke as illuminated by a beam of parallel rays after passing through the lens as indicated by the arrows in the figure. The outline of the lens can be seen faintly at the left in the photograph which is entirely untouched except for the arrows and the lettering.

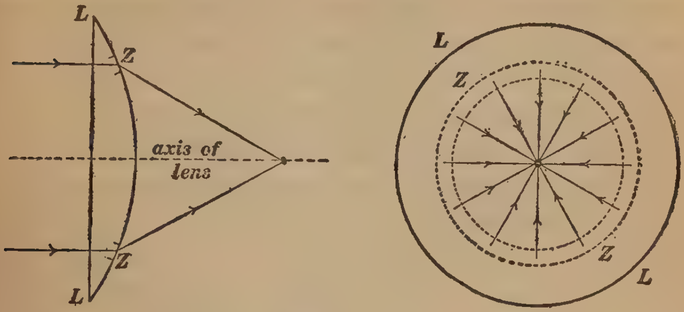

Fig. 113.

The light which passes through the middle part of the lens is focussed at *F*, the light which passes through the edge of the lens is focussed at *F′*, and the light which passes through the intermediate parts or zones of the lens is

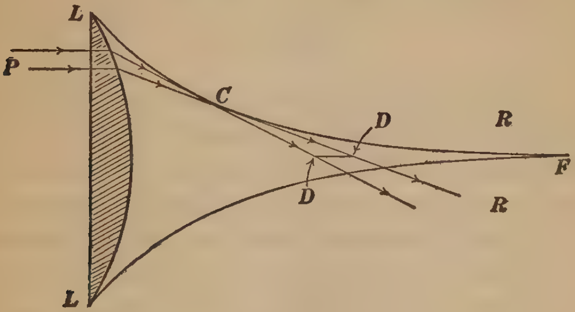

Fig. 114.

focussed along the bright line between *F* and *F′*. To fully understand the details of Fig. 112 let us consider the sketches which are shown in Figs. 113 and 114. Figure 113 shows the rays which pass through the zone

ZZ of the lens (the zone which lies between the dotted circles in Fig. 113); and Fig. 114 shows a side view of a narrow pencil of rays *PCD*. This pencil becomes an astigmatic pencil after passing through the lens; one focal line of the astigmatic pencil is at *C* (perpendicular to the plane of the paper) and the other focal line of the pencil is the short portion *DD* of the axis of the lens. The curved line in Fig. 114 (which, of course, represents a surface) is a caustic, and it is visible in Fig. 112 as the sharp boundary of the illuminated cloud of smoke.

72. Astigmatism. A narrow pencil of parallel rays, or indeed, any narrow homocentric pencil of rays, becomes an astigmatic pencil when it passes obliquely through a simple lens. Thus, Fig. 115 shows the focal lines *C* and *DD* of the astigmatic pencil of rays which is produced by the passage of the narrow pencil *rr* obliquely through the lens, as shown. The focal point of the lens is at *F*. This conversion of a homocentric pencil of rays into an astigmatic pencil by oblique passage through a lens is called *astigmatism*, and a compound lens which is free or approximately free from astigmatism is called an *anastigmatic*, or simply a *stigmatic lens*.

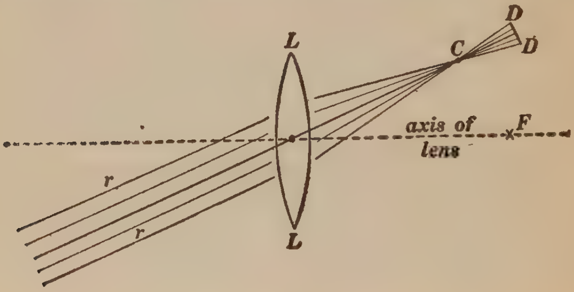

Fig. 115.

A very simple demonstration of the astigmatism of a simple lens may be made by looking obliquely through a magnifying-glass at the cross-rulings on a sheet of cross-section paper. Under these conditions one or the other set of cross-rulings may be sharply focussed according to the distance of the lens from the paper.

A more striking demonstration of astigmatism may be made by projecting a cross-ruled lantern slide upon the screen with the lantern objective turned so that the light passes through it obliquely. For this experiment use a simple lens as the lantern objective and cover all but the central portion of the lens with opaque paper.

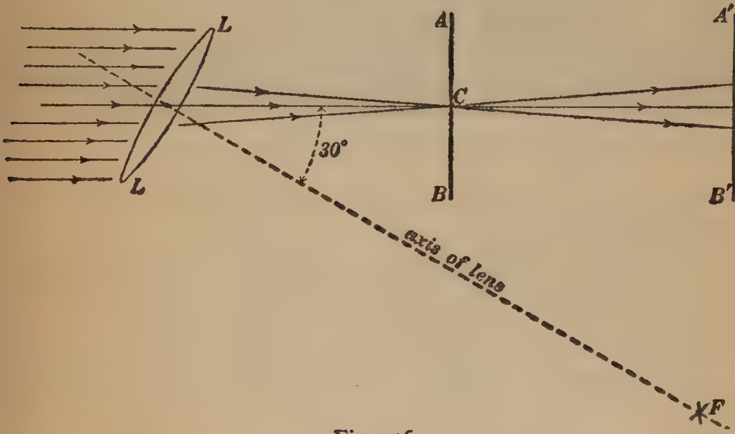

Fig. 116.

Astigmatism of the eye is due to unequal curvatures of the refracting surfaces of the eye in different directions, so that a homocentric pencil of rays which enters the eye *axially* becomes an astigmatic pencil. This kind of astigmatism must not be confused with the astigmatism of a lens which is the conversion of an *oblique* homocentric pencil into an astigmatic pencil.

73. Oblique spherical aberration of a broad beam or pencil.

Coma. Figure 116 is a sketch, exactly one quarter size, showing a beam of parallel rays passing obliquely through a simple converging lens *LL*. Figure 117*a* is a photograph full size of the spot of light on the plate *AB* of Fig. 116, and Fig. 117*b* is a photograph of the spot of light on the plate *A′B′* of Fig. 116, the full opening of the lens *LL* being used.

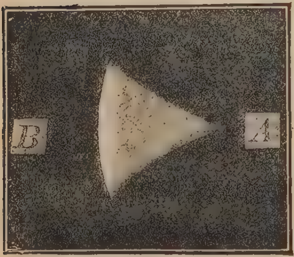

Fig. 117*a*.

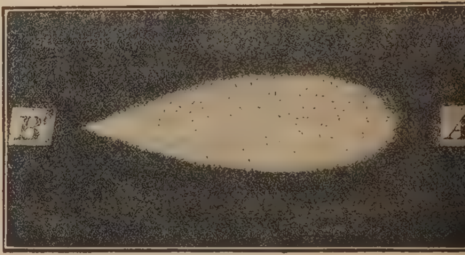

Fig. 117*b*.

When the central zone only of the lens *LL* is used, the astigmatic pencil is focussed along a horizontal line at *C* (perpendicular to the plane of the paper) and along the vertical line *DD* in Fig. 116, as shown by the two sketches in Fig. 118. In these sketches the outline of the coma is also shown for the sake of comparison.

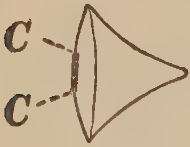

Fig. 118*a*.

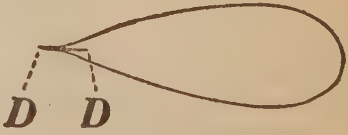

Fig. 118*b*.

74. Compensation for spherical aberration. Aplanatism. Lens imperfections are always eliminated (approximately) by balancing, as it were, the opposite imperfections of two or more simple lenses which are used together as a compound lens.* The possibility of eliminating axial spherical aberration in this way was discovered about 1760, and the first step towards the correcting of a compound lens for oblique spherical aberration was made by Frauenhofer about 1810. A more complete solution of the problem of oblique spherical aberration was made by Abbe in 1873, when he discovered what is known as *Abbe's sine condition* which refers to slightly oblique rays passing through a very wide aperture lens. The possibility of correcting a compound lens for astigmatism, which refers to very oblique rays through a lens of comparatively narrow aperture, was discovered by Rudolph about 1890.†

Many of the earlier lens manufacturers had learned by trial to satisfy Abbe's sine condition, but since the discoveries of Abbe and Rudolph, the lens designer recognizes more distinctly the necessity and understands more clearly the possibility of eliminating astigmatism and coma, but the lens designer must still work very largely by trial, because of the infinite number of possible combinations of differently shaped lenses of different kinds of glass at different distances apart.

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* This is true except in the case of very large telescope objectives, where it is only partially true. Some of the imperfections of a large telescope objective are eliminated by bringing one or more of the lens surfaces to a desired shape (other than spherical) by very tedious local polishing.

† An example of the complete calculation of a telescope object glass, according to Gauss, is given by Otto Lummer in Müller-Pouillet's *Lehrbuch der Physik*, Vol. II, Part I, pages 573–579. The method of eliminating spherical aberration is brought out in this example.

The theory of the aberrations of a lens was worked out very completely by von Seidel in 1855. A good discussion of von Seidel's theory may be found in Lummer's *Photographic Optics*, translated by S. P. Thompson, pages 6–13, and pages 103–115; and a good discussion of Abbe's sine condition may be found on pages 116–121. See also Drude's *Theory of Optics* translated by Mann and Millikan, pages 58–63. An extremely simple discussion of the five aberrations of von Seidel on the basis of Hamilton's Principle is given by Lord Rayleigh in the *Philosophical Magazine* for June, 1908, pages 677–687.

A lens which is free from spherical aberration is said to be *aplanatic*. This term was originally applied to a lens which had been corrected only for axial spherical aberration, but nowadays we must distinguish two cases as follows :

(*a*) A very wide aperture lens may be corrected for axial spherical aberration and for the aberration of a very slightly oblique broad beam. Such a lens is said to be *aplanatic* ; but it is aplanatic only for one particular pair of conjugate points which are called the *aplanatic points* of the lens.

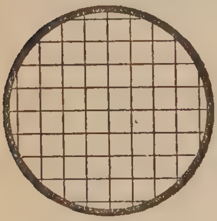

Fig. 119.

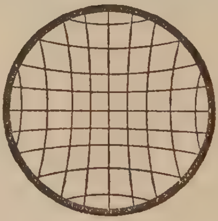

Fig. 120.

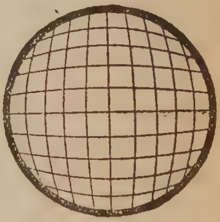

Fig. 121.

(*b*) A narrow aperture lens may be corrected for the spherical aberration of a very oblique narrow pencil (astigmatism). Such a lens is said to be *anastigmatic*.

75. Image distortion. The image of an object formed by a lens is in general distorted. Thus, Fig. 119 represents a square network of lines, and Figs. 120 and 121 show two distorted images of this network. In Fig. 120 the magnification of the image is greater near the edges of the field of view than it is near the center, and in Fig. 121 the magnification of the image is less near the edges of the field of view than it is near the center.

The distortion of an image by a lens may be shown in a very striking way as follows: A magic lantern is provided with a simple objective lens *LL*, Figs. 122 and 123, and the image of a cross-ruled lantern slide *SS* is projected upon the screen.

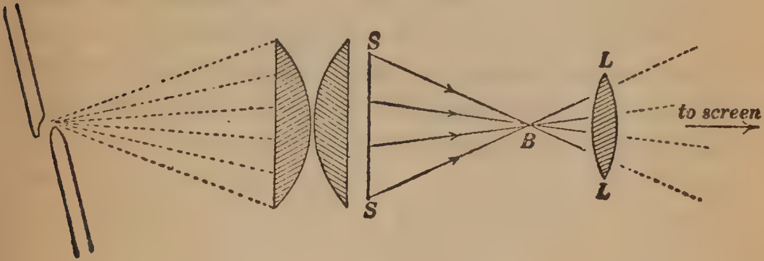

Fig. 122.

(*a*) The source of light in the lantern is adjusted so that the light, after passing through *SS*, is concentrated at *B*, Fig. 122, *as if the point B were a small hole in a diaphragm in front of the object lens LL*. In this case the cross-rulings, as projected on the screen, appear like Fig. 120.

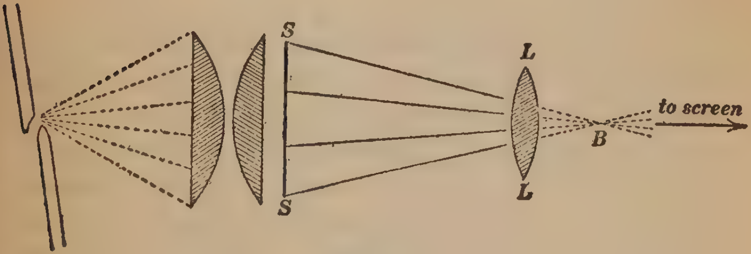

Fig. 123.

(*b*) The source of light in the lantern is then adjusted so that the light, after passing through *SS*, is concentrated at the point *B*, Fig. 123, *as if the point B were a small hole in a diaphragm behind the object lens LL*. In this case the cross-rulings, as projected on the screen, appear like Fig. 121.

These experiments show that the distortion of an image is of one kind or the other (like Fig. 120 or Fig. 121) according as the diaphragm, or stop, *B* is in front of or behind the lens, and one is able to see in a general way that one kind of distortion may be compensated by the other kind by placing the diaphragm between two lenses of a system, as shown in Fig. 124, in which the diaphragm, or stop, *DD* is behind one lens *LL*, and in front of the other lens *L′L′*.

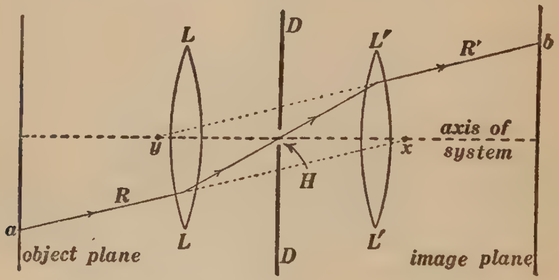

Fig. 124.

The action of the symmetrical double-lens of Fig. 124 may be understood from the following considerations. The hole *H* in the diaphragm is assumed to be small so that *all* incident rays like *R* which come from points in the object plane intersect at the point x which is conjugate to the point *H* with respect to the front lens *LL*, and *all* emergent rays like *R′* intersect at the point y which is conjugate to the point *H* with respect to the back lens *L′L′*. Furthermore, every incident ray *R* is parallel to the corresponding emergent ray *R′*. Therefore the points like *b* in the image plane as seen from the point y are distributed in exactly the same way as the corresponding points like *a* in the object plane as seen from the point *x*, or, in other words, the image is exactly similar to the object.*

A lens system which gives an undistorted image of an object is called a *rectilinear* or *orthoscopic* system. Such lenses are always used for photo-engraving, where it is desired to produce an accurate copy of a drawing, and for photographing buildings.

76. Curvature of field. In order to project on a screen the most distinct image of an extended object which it is possible to form by means of a simple lens, the screen must be curved as shown in Fig. 125. It is instructive to note the relation between Figs. 115 and 125. There is, between *C* and *DD* in Fig. 115, a place where the astigmatic pencil would produce the smallest possible luminous spot, the circle of least confusion, so-called, and the screen *SS* in Fig. 125 must pass between *C* and *DD* of Fig. 115.

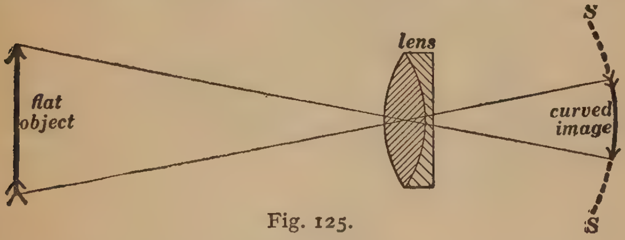

Fig. 125.

The imperfection of a lens which is here described is called *curvature of field*, and a lens system in which this error is corrected is said to have a *flat field*.†

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* This discussion ignores the spherical aberration of the lenses *LL* and *L′L′* in Fig. 124. The theory of the orthoscopic lens is quite fully discussed on pages 29–39 of Lummer's *Photographic Optics*, translated by S. P. Thompson.

† Curvature of field is intimately connected with astigmatism, although von Seidel's theory gives two distinct conditions, one for the elimination of astigmatism and another for the elimination of curvature of field. See Winkelmann's *Handbuch der Physik*, Vol. VI, pages 139–143. Curvature of field is discussed on pages 10, 23, 57, and 61–67 of Lummer's *Photographic Optics*.

77. Chromatic aberration. The five lens errors, axial spherical aberration, astigmatism, coma, image distortion and curvature of field, refer to the action of a lens when *light of one wave-length* (one color) is used. The use of white light introduces another complicated imperfection which is called *chromatic aberration* and which is due to the fact that a given sample of glass has different refractive indices for different wave-lengths (colors) of light. Thus, Fig. 126 shows the focal points *v* and *r* of a lens *LL* for violet light and for red light respectively; the focal points for the other colors lie between *v* and *r*.

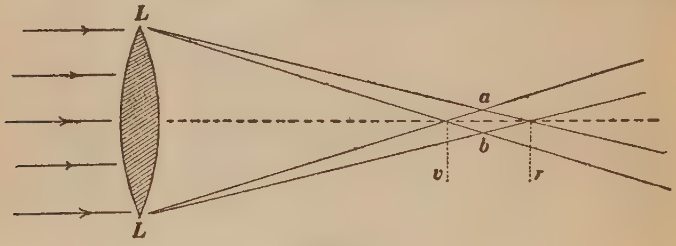

Fig. 126.

The smallest diameter of the focal spot *ab*, Fig. 126, is about one thirty-third of the diameter of the lens.

The distance *vr*, Fig. 126, for a lens of given focal length varies greatly with different kinds of glass, and it is therefore possible to construct a converging lens of one kind of glass and a diverging lens of another kind of glass so that when the two lenses are used together as a compound lens system the chromatic aberration of one of the lenses is annulled (nearly) by the opposite chromatic aberration of the other, while the converging action of the one is not annulled by the diverging action of the other. A compound lens which is compensated for chromatic aberration in this way is said to be *achromatic*. The achromatic doublet was devised in 1758 by Dolland, who used such doublets for the object-glasses of his telescopes. A sketch of the theory of the achromatic doublet is given in the chapter on dispersion.

Some idea of the complexity* of chromatic aberration may be obtained by the following discussion of Figs. 127, 128, and 129. A lens system may be "achromatized" (*a*) so as to *form images of all colors*† *in one plane*, or (*b*) so as to *provide for equal-sized images of all colors*.† In the first case the different colored images will be of different sizes, and in the second case the different colored images will be in different planes. The first is called the *achromatization of the focal plane*‡ and the second is called the *achromatization of magnification*.§

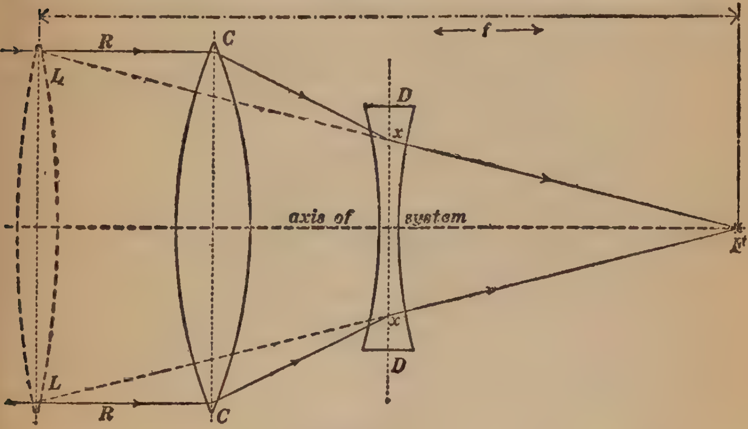

Fig. 127.

Consider a converging lens *CC* and a diverging lens *DD* arranged as shown in Fig. 127. Let us for a moment ignore chromatic aberration and consider that the two lenses *CC* and *DD* focus a beam of parallel rays at the point *F*. Extend the two rays *Fx* backwards until they intersect the incident rays *RR* at *LL*. *The combined action of the two lenses CC and DD is equivalent to a single lens at LL of which the focal length is equal to f as shown.*

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* This statement does not refer to the impossibility of compensating chromatic aberration for all colors.

† Strictly, two colors only.

‡ Sometimes called *achromatization of the focal point*.

§ Sometimes called *achromatization of the focal length*.

Let *CC*, Fig. 128, represent a converging lens of crown glass and *DD* a diverging lens of flint glass, the two lenses being designed to bring both red light r and violet light v to a focus at the point F as shown. The two lenses *CC* and *DD*, in so far as their action on red light is concerned, are together equivalent to a single lens at A of which the focal length is f_r as shown; and the two lenses are equivalent to a single lens at B (focal length f_v) in so far as their action on violet light is concerned. The lens system *CCDD* of Fig. 128 produces red and violet images of a distant object in the same plane, namely, the plane containing the point F, but the violet image is larger than the red image in the ratio of f_v to f_r. The lens system *CCDD*, Fig. 128, is achromatized for focal point but not achromatized for focal length.

Figure 129 represents a combination of a converging crown glass lens and a diverging flint glass lens which is achromatized for focal length ($f_r = f_v$). Such a system would give the same sized violet and red images of a distant object, but the images would not be in the same plane. In fact the red image would be in the plane containing F_r and the violet image would be in the plane containing F_v.

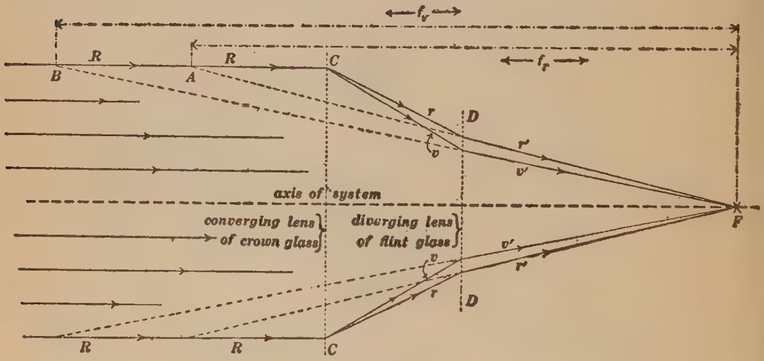

Fig. 128.

When the two lenses *CC* and *DD* are very near together (near, that is, in comparison with their focal lengths), then achromatization of the focal point carries with it the achromatization of focal length, as is evident from a careful study of Figs. 128 and 129. Complete* achromatization is therefore much more easily accomplished when the individual lenses of a system are near together than when they are far apart. In photographic objectives it is usually desirable to separate the individual lenses, and in such cases *achromatic doublets* are used instead of simple individual lenses.

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* Complete in the sense of including achromatization of focal point and achromatization of focal length, not in the sense of being achromatized for all colors of the spectrum.

A lens system which consists of two lenses of similar glass at a distance apart equal to half the sum of their individual focal lengths has *the same focal length for all wave-lengths* (all colors). In this case, however, the lenses are at a great distance apart as compared with their focal lengths, and although such combinations have the same focal length for all wave-lengths, they do not have the same focal point for all wave-lengths. Such a combination is therefore achromatized for focal length but not achromatized for the focal point. The doublets of Huygens and Ramsden which are so much used for eye-pieces for telescopes and microscopes are examples of partially achromatized systems of this type.

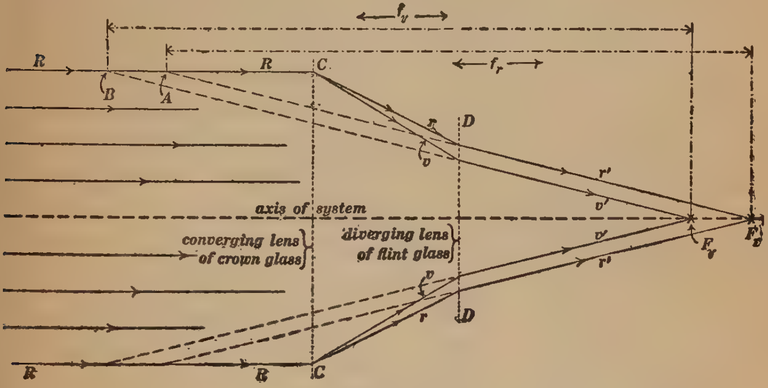

Fig. 129.

Chromatic differences of spherical aberration. A lens system may be accurately aplanatic (with respect to its aplanatic points, of course) for a given wave-length (color) of light and non-aplanatic for other wave-lengths (colors). This error of a lens is called the *chromatic difference of spherical aberration.*

78. The wide-angle lens. A lens in the form of a complete sphere having a diaphragm *DD* with a small hole at its center, as shown in Fig. 130, has a field angle of nearly 180°, but the field is very strongly curved, as shown. The wide-angle photographic objective involves the principle of the spherical lens, modified more or less to give flatness of field. This is exemplified by Fig. 139, which is a moderately wide-angle photographic objective, and by Fig. 142, which is an extremely wide-angle photographic objective.

79. Wide-angle lenses versus wide-aperture lenses. A lens cannot be made to give a wide field-angle and to have at the same time a large numerical aperture. The combination of these two things is impracticable. *When a very wide field-angle is desired one must be content with small aperture, and when a very wide aperture is desired one must be content with small field-angle.* Thus, a high-grade microscope objective with a numerical aperture of about 1·4 has a field-angle of not more than half a degree, and a wide-angle photographic lens having a field-angle 110 degrees has a numerical aperture of about 1/36.

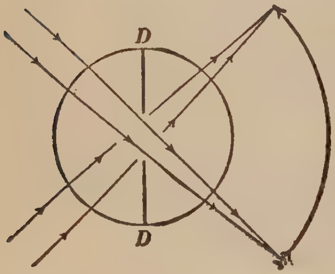

Fig. 130.

There is a demand in photography, however, for lenses having a moderately wide aperture and giving a moderately wide field, and many photographic lenses are now available which give fairly good definition over a field from 30 to 60 degrees wide with numerical aperture ranging from 1/4 to 1/8. See Figs. 140 and 141.

A wide-aperture lens must be, above all things, compensated for spherical aberration and for chromatic aberration. The effect of spherical aberration in producing very great blurring at the focus of a large-aperture lens as compared with the blurring at the focus of a small-aperture lens is shown in Fig. 131. The same effect for chromatic aberration is shown in Fig. 132. In these figures *SS* represents the position of the screen for which the focal spot is the smallest possible. The distance *ab* in Fig. 131 is roughly proportional to the square of the aperture of the lens so that *a′b′* in Fig. 131 is very much less than *ab*, and the size of the focal spot is extremely small for lens *B* as compared with its size for lens A. The distance *rv* in Fig. 132 is independent of the aperture of the lens, but the figure shows nevertheless that a beam of light is focussed in a much smaller spot by lens *B′* than by lens *A′*.

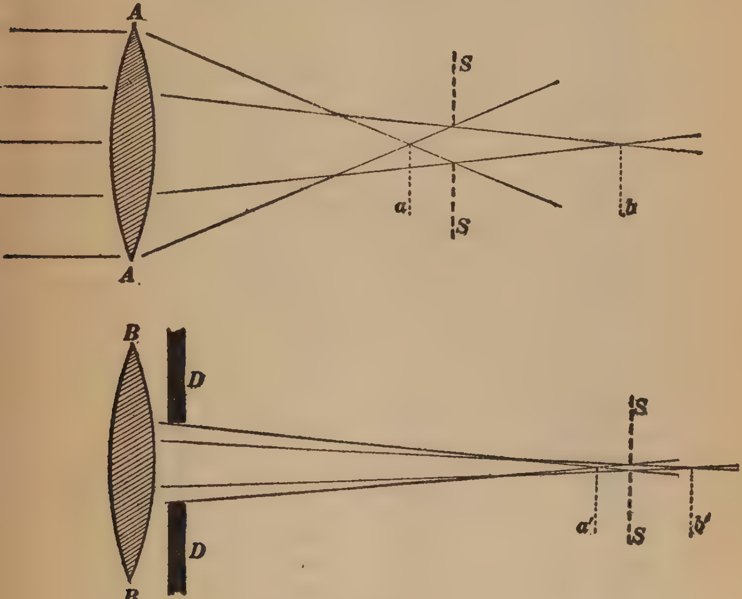

Fig. 131.

A wide-angle lens must be, above all things, compensated for astigmatism, distortion, and curvature of field. Most wide-angle lenses depend, however, upon narrowness of aperture for keeping these imperfections within practicable limits. This is exemplified by Figs. 139 and 142.

80. Examples of compensated lens systems. (*a*) *Tele-*

scope objectives. The telescope objective is generally used where a *small field-angle* is desired, and the ordinary telescope objective is compensated only for spherical aberration and for chromatic aberration. Figure 133 shows a sectional view, actual size, of an aplanatic achromatic telescope object-lens of 18 inches focal length as designed by Fraunhofer. It consists of a double convex crown glass lens *CC* and a concavo-convex flint glass lens *FF*. Figure 134 is a sectional view, actual size, of a modern high-grade opera-glass of which the object-lens *O* and the eye-lens *E* are triplets. The object-lens of an opera-glass is much larger in numerical aperture (larger in diameter for a given focal length) than the object-lens

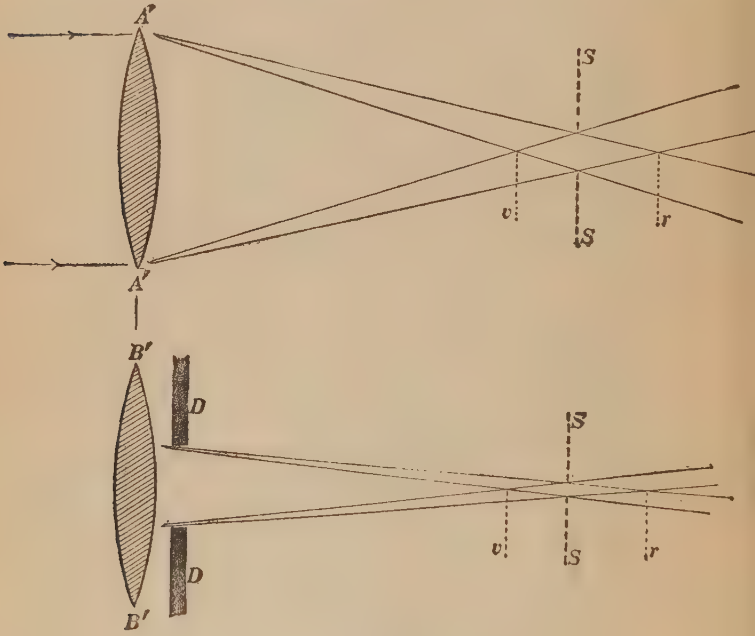

Fig. 132.

of the astronomical telescope as usually constructed. It is for this reason desirable to use three lenses for the object-lens of an opera-glass, as shown in Fig. 134, in order that the various imperfections may be more completely compensated than would be possible by the use of two lenses, as shown in Fig. 133.

(*b*) *Eye-pieces.** Figure 135 shows an eye-piece doublet which was designed by Huygens about 1680. It is partially achromatic (achromatized for focal length but not for focal points) and it has a wide flat field. It is extensively used at the present day with compound microscopes and telescopes where it is not desired to use cross-wires in the image plane of the object-glass.

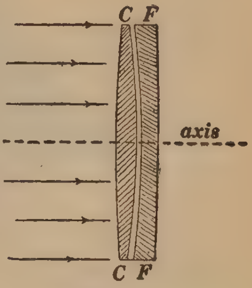

Fig. 133.

Figure 136 shows an eye-piece doublet which was designed by Ramsden in 1783. It is partially achromatic (achromatized for focal length but not for focal points) and it has a wide flat field. It is extensively

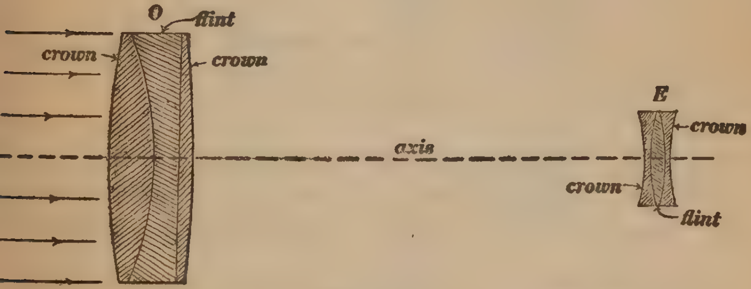

Fig. 134.

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* A simple discussion of eye-pieces is given in Edser's *Light for Advanced Students*, pages 204–212.

used at the present day with compound microscopes and telescopes where it is desired to use cross-wires in the image plane of the object-glass as shown in Fig. 105.

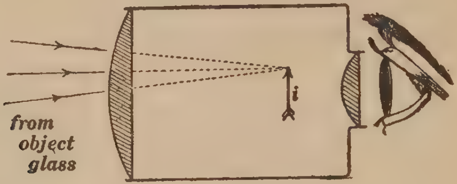

Fig. 135.
Huygens eye-piece.

Figure 137 shows a highly perfected achromatic and orthoscopic eye-piece designed by Abbe.

(*c*) *Photographic lenses.** The first step in the development of the modern high-grade photographic lens was taken by Wollaston in 1812, who substituted the meniscus lens (see Fig. 56) for the simple bi-convex lens which was used in the camera obscura† before his time, and he speci-

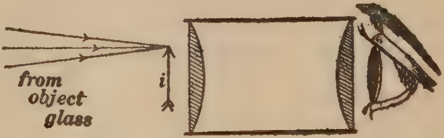

Fig. 136.
Ramsden eye-piece.

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* A good outline of photographic optics is given by Lummer in the *Zeitschrift für Instrumentenkunde*, 1897, pages 208, 225, and 264. These articles have been translated into English by S. P. Thompson and published as *Photographic Optics* by Macmillan & Co., 1900. A very complete discussion of the theory and history of the photographic objective is von Rohr's *Theorie und Geschichte des Photographischen Objectivs*, Berlin, 1899. See also *Photography for Students* by Louis Derr, The Macmillan Company, 1906.

† The camera obscura is a dark chamber, or box, with a lens at one side for projecting an image of an external object or landscape.

fied a particular position for the diaphragm or stop. The simple meniscus lens is extensively used at the present day in cheap forms of photographic cameras.

The achromatic doublet came into use for the camera obscura about 1835. This lens has been and is still extensively used for photographic purposes.

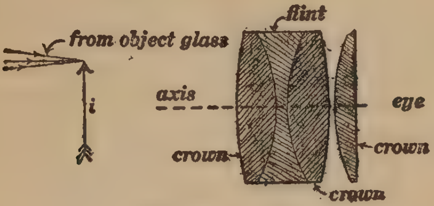

Fig. 137.
Abbe eye-piece.

The demand for a quick-acting lens (wide numerical aperture) which came with the invention of photography was met by the remarkable portrait lens of J. Petzval which was designed in 1840 and manufactured by Voigtländer. Figure 138 shows a sectional view, full size, of a Petzval portrait objective having a focal length of 10 centimeters. This lens has a numerical aperture of 1/3·5

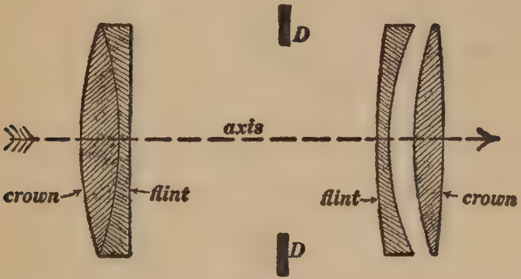

Fig. 138.
Petzval's portrait lens.

and it is accurately aplanatic. It gives extremely good definition in the center of its field and fairly good definition over a field of about 25 or 30 degrees. This objective and the portrait objective of J. H. Dallmeyer, which was brought out in 1866, are still extensively used.

After the remarkable achievement of Petzval, the next notable improvement in the photographic lens was made in England, and the first symmetrical orthoscopic lens was the "Globe lens" of Harrison and Schnitzer, which was brought out about 1860. Figure 139 is a sectional view, full size, of a "Globe lens" of 10 centimeters focal length.

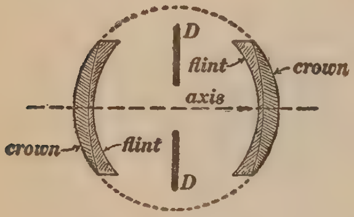

Fig. 139.
The "Globe lens."

In 1866 the symmetrical aplanatic orthoscopic lens of Steinheil was produced. A sectional view, full size, of one of these lenses of 20 centimeters focal length, is shown in Fig. 140.

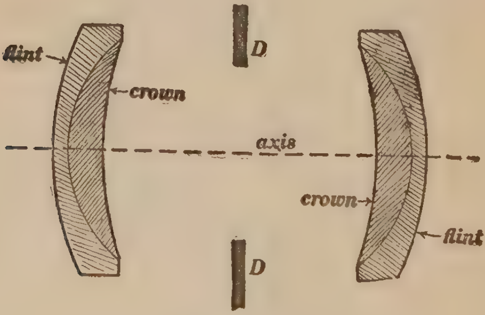

Fig. 140.
Steinheil's lens.

All converging lenses in Figs. 138, 139, and 140 are of crown glass and all diverging lenses are of flint glass. The compensation of lens errors depends upon the combination of lenses made of different kinds of glass, and the development of photographic lenses (and also of microscope objectives) was greatly stimulated after the establishment of the celebrated Jena Glass Works in 1885 for the manufacture of a variety of new optical glasses under the direction of Abbe.

Figure 141 shows one of the best of present-day photographic lenses with an aperture of 1/4·5. It is very completely corrected for axial and oblique spherical aberration, it is very nearly orthoscopic and it gives a flat field.

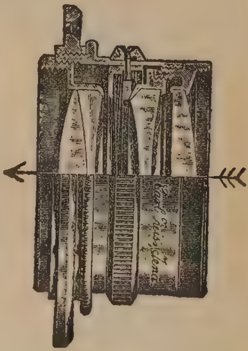

Fig. 141.

"Tessar lens." Carl Zeiss. Patented 1902 by P. Rudolph.

The symmetrical achromatic doublet which is shown in Fig. 139 gives good definition over a field of 90° or more, but Fig. 142 shows an extremely wide-angle lens which gives good definition (with a small stop) over a field of 135°. This lens is interesting in that it illustrates the fact that a very narrow aperture lens does not need especially to be corrected for spherical and chromatic aberration as explained in connection with Figs. 131 and 132. The two lenses which are combined in Fig. 142 are simple meniscus lenses and they are not achromatic.

For making photographs of a distant small object, a very long focal length lens is necessary if a fair-sized photograph is to be produced. In order to avoid the great inconvenience which would be involved in the use of a very long focal length lens of the ordinary type, a combination called the *telephotographic lens* is used. Figure 143 shows a telephotographic lens. The equivalent focal length of this lens may be varied by changing the distance between the front and back combinations, and one may easily obtain an equivalent focal length of 8 or 10 feet with a camera length of 2 or 3 feet. The action of the telephotographic combination may be understood with the help of Fig. 127.

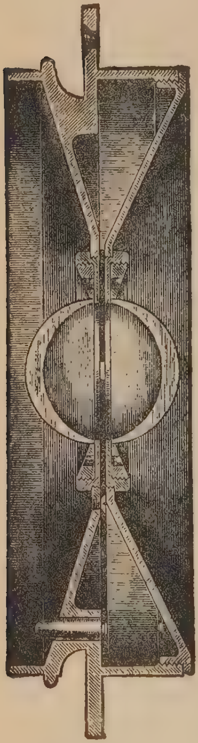

Fig. 142.
"Hypergon lens." Patented 1900 by C. P. Goerz.

(*d*) *Microscope objectives.* Low-power microscope objectives usually consist of one achromatic doublet. Figure 144 shows a low-power microscope objective consisting of two achromatic doublets combined to form one system.

In high-power microscope objectives, however, the front lens of the object-glass (*the lens nearest to the object*) *is always made in the form of a hemisphere, and in the most powerful microscope objectives the space between the front lens and the object which is being examined is filled with oil* (*the oil having the same index of refraction as the front lens of the object-glass*) *as shown in Fig.* 145. A microscope objective when so used is called an *oil immersion objective.* The front lens in Fig. 145 is, under the specified conditions, completely free from axial spherical aberration as explained under Case II of Art. 45.

Figure 145 shows (three times actual size) a highly perfected microscope objective of 2 millimeters (one-twelfth inch) focal length. This objective was designed about 1886 by Abbe and it is known as the *apochromatic objective.* It has the largest possible numerical aperture, and, although it is corrected only for spherical aberration and for chromatic aberration, it involves eleven separate compensating effects. It is achromatized for focal point for three colors, and its spherical aberration is eliminated for each of two colors for slightly oblique rays as well as for rays parallel to the axis. The lack of achromatization of focal length causes the objective to form colored images of different sizes, and this imperfection is compensated by using a specially designed non-achromatic eye-piece which

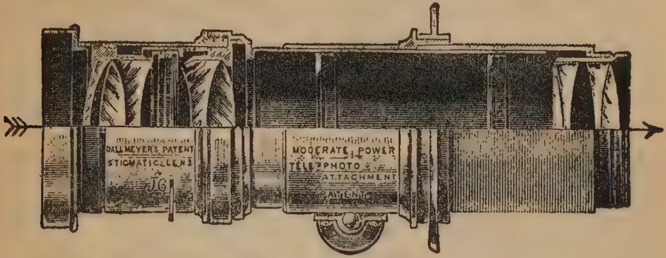

Fig. 143.
" Telephoto lens." Patented about 1891 by T. R. Dallmeyer.

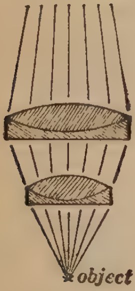

Fig. 144.

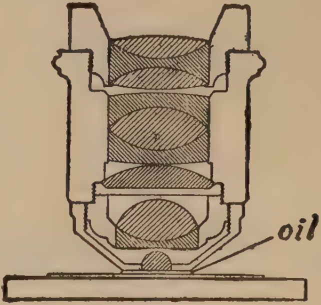

Fig. 145.

magnifies the different colored images differently, thus giving a single colorless resultant image. The excellence of this objective is such that the absolute limit of effective magnifying power as fixed by the nature of light (as explained briefly in Art. 67) may be considered as actually attained by it.

LEADING QUESTIONS

L. Q. 42. How can a lens be perfect and yet not bring a beam of parallel rays to a point focus ?

L. Q. 43. Under what conditions is a wide numerical aperture lens desirable ? Under what conditions is a narrow numerical aperature lens necessary ?

L. Q. 44. What is meant by an aplanatic lens ? What is meant by the aplanatic points of an aplanatic lens ?

L. Q. 45. What is an anastigmatic lens ?

L. Q. 46. What is an orthoscopic or rectilinear lens ?

L. Q. 47. Why should a wide-aperature lens be, above all things, corrected for spherical and chromatic aberration ?

L. Q. 48. What imperfections are most serious in a narrow aperature lens when used with a wide field angle ?

L. Q. 49. Why is the lens system in Fig. 128 said to be achromatized for focal point but not achromatized for focal length ?

PROBLEMS

62. Let us suppose that the desired degree of sharpness of a photograph is obtained when no focal spot on the plate is more than 0·01 inch in diameter. A camera whose lens has a focal length of 6 inches is accurately focussed on an object which is 60 inches from the center of the lens. Find, for this adjustment of the camera, the distance from the center of the lens to the nearest object which will be focussed to a satisfactory degree of sharpness ; (*a*) When the numerical aperture of the lens is 1/16, and (*b*) When the numerical aperture of the lens is 1/8. Ans. (*a*) 48·4 inches; (*b*) 53·6 inches.

Note. Take no account of diffraction in this problem.

63. Solve problem 62 for a lens whose focal length is 12 inches and whose numerical aperture is 1/16 and 1/8.

64. A photographic lens with a numerical aperture of 1/6 gives a good photograph with an exposure of 0·02 second. What exposure would be required with a numerical aperture of 1/16?

65. The angular diameter of the sun is 0·5 degree. Find the diameter of the image of the sun formed: (*a*) by a lens whose focal length is 2 inches, and (*b*) by a lens whose focal length is 20 inches, (*c*) the two lenses have the same diameter, and the heat energy delivered per second to each unit of area of the image formed by the lens whose $f = 2$ inches is h times as great as the heat energy delivered per second to each unit of area of the image formed by the lens whose $f = 20$ inches. Find h.

66. An apochromatic microscopic objective, without the oil-immersion feature which is shown in Fig. 145, has a numerical aperture of 1·4. Ignoring the excessive loss of light by reflection from the numerous lens surfaces in Fig. 145 and by absorption in the great thickness of glass traversed, find how much exposure would be required with this apochromatic objective when a "Tessar" lens (see Fig. 141) whose numerical aperature is 1/4·5 requires 1/200 of a second.

Note. If a large-model apochromatic objective were to be used for ordinary photography the hemispherical lens in Fig. 145 would face towards the photographic plate.

CHAPTER VIII

DISPERSION.* SPECTRUM ANALYSIS †

81. Newton's experiment.‡ Homogeneous light. Non-homogeneous light. A beam of parallel rays of white light, such as sun light or lamp light, is changed into a fan-like beam by passage through a prism. Thus, the beam of parallel rays B, Fig. 146, is changed into the fan-like beam

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* A good discussion of dispersion is given in Edser's *Light for Students*, pages 375–387. See also Wood's *Physical Optics*, pages 85–99 and 308–348. The modern theory of dispersion is given in Drude's *Theory of Optics* (translated by Mann and Millikan), pages 382–399.

The *rainbow* is produced by the dispersion of sun light in drops of rain. Other interesting optical phenomena of the atmosphere are *mirage, coronas and halos, scintillation* and the *color of the sky*. These various matters are discussed in Hastings' *Light*, pages 111–153 (Scribner's, 1901); Preston's *Theory of Light*, pages 529–541 (Macmillan & Company, 1901); Edser's *Light for Students*, pages 101–107 (Macmillan & Company, 1902); and Wood's *Physical Optics* (The Macmillan Co., 1905), pages 69–78. Wood's discussion of mirage is especially interesting. The sharp-edged appearance of the sun is a mirage effect. See A. Schmidt, *Physikalische Zeitschrift*, Vol. IV, pages 282–285, February, 1903.

† A very good simple discussion of spectrum analysis and its teachings is to be found on pages 330–360 of Edser's *Light for Students*, Macmillan, 1902. One of the best small works in English on spectrum analysis is *Spectroscopy* by E. C. C. Baly (Longmans, Green & Co.). A very interesting discussion of the application of the interferometer to spectroscopy is given on pages 60–83 of Michelson's *Light Waves and their Uses* (University of Chicago Press, 1903).

The most complete old work is Kayser's *Spectralanalyse*, and perhaps the best résumé is Kayser's chapter (pages 654–784 of Volume VI) in Winkelmann's *Handbuch der Physik*.

‡ An interesting account of Newton's original experiments with the prism may be found in Preston's *Theory of Light*, pages 117–122.

B' by passage through the prism P. This fan-like beam in falling upon a screen SS produces an illuminated band RV called a *spectrum* which is red at the end R and passes by insensible gradations through orange, yellow, green, and blue to violet at the end V. The beam B of white light is said to be *dispersed* by the prism. A photographic plate reveals the existence of invisible rays beyond V. These rays are called the *ultra-violet* rays. A heat detector shows the existence of rays below R. These rays are called *infra-red* rays. The portion of the spectrum between R and V is called the *visible spectrum.*

The beam of white light B, Fig. 146, is deflected by the prism and also spread out or dispersed. On the other hand, the beam of light B'', which passes through a small hole in the screen, can be deflected by a prism but it cannot be spread out or dispersed.

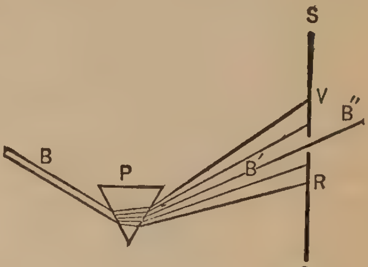

Fig. 146.

Since 1912 such enormous advances have been made in the theory of spectra by the application of the quantum hypothesis combined with the modern theories of atomic structure, that all older books on spectrum analysis are out of date.

A very good discussion of some of the more recent developments in this field is given by Norman Campbell in Chapter XV of his *Modern Electrical Theory*. This chapter is a supplement to the original book, and it is published by the Cambridge University Press, 1921.

A good discussion of the earlier phases of Bohr's theory of the emission of light is given by J. H. Jeans in a *Report on Radiation and the Quantum Theory*, The London Physical Society, 1914.

A full discussion of the theory of spectra is given in a recent book entitled *The Origin of Spectra*, by Paul D. Foote and F. L. Mohler, American Chemical Society Monograph No. 8, September, 1922.

A very important book is Sommerfeld's *Atomic Structure and Spectral Lines* (translated by H. L. Brose) Methuen & Co., London, 1923.

The beam of white light is evidently made up of dissimilar parts, because these parts are unequally deflected by a prism. Therefore, white light is called *non-homogeneous* light. The beam B'', on the other hand, consists of but one kind of light, because it is deflected by the prism without being dispersed. Therefore, the beam B'' is called *homogeneous** *light.* Since the prism P deflects the different parts of the non-homogeneous beam B differently, it is obvious that the glass of which the prism is made has a different index of refraction for each of the homogeneous parts or components of white light.

The phenomena of interference which are described in the next chapter show that *a beam of homogeneous or monochromatic light is a simple wave-train of definite wave-length* or, more strictly speaking, *a simple wave-train of definite frequency*, inasmuch as the wave-length is halved, for example, when a given wave-train passes into a medium in which the velocity is halved. *A beam of white light, on the other hand, is an utterly irregular succession of wave-pulses and short sections of wave-trains of every variety of wave-length.* In a vacuum, all these different waves travel at the same velocity, and this is approximately true in air also. In substances like glass, wave-trains of different wave-lengths have distinctly different velocities, and therefore the different wave-trains which make up a beam of non-homogeneous light are differently refracted by a glass prism.†

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* Homogeneous light is sometimes called *monochromatic light*, or light of one color.

† The action of a prism in resolving a beam of white light into a series of simple wave-trains is a phenomenon of resonance. See Drude's *Theory of Light* (translated by Mann & Millikan), pages 382–399.

The velocity of ordinary water waves, especially of the variety known as ripples, varies greatly with the wave-length, and a phenomenon which is closely analogous to the dispersion of light by a prism (due to the different velocities of the wave-trains of light of different wave-lengths) is the following: An oar is dipped gently into the smooth surface of a pond and the irregular wave which is produced by the oar is quickly resolved into a series of fine ripples, as shown at *A*, Fig. 147. The fine ripples of short wave-length travel faster than the coarse ripples of long wave-length and are thus separated from them, as shown.

82. The spectroscope. In the spectrum as obtained by Newton (see Fig. 146), the beam of white light has some breadth and the various beams of homogeneous light into which the white light is resolved by the prism are each as wide as the original beam of white light. Therefore the various beams of homogeneous light overlap each other greatly, that is to say, each point on the screen in Fig. 146 is illuminated by several overlapping beams of homogeneous light. The *spectroscope* is an instrument for separating as completely as may be the homogeneous components of a beam of non-homogeneous light.

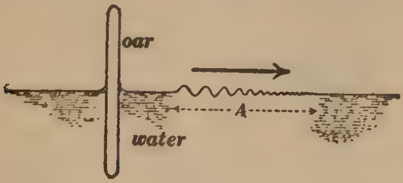

Fig. 147.

The spectroscope is exemplified in its simplest form by placing a large prism *P* in front of the object-glass *O* of a telescope *OE*, as shown in Fig. 148; the light from a star is then deflected by the prism and appears to have come from *X*. In fact, the different wave-lengths of light from the star are differently deflected, and the light appears to have come from a number of stars near *X*, one star for each wave-length of light. The result is that the object-glass *O* of the telescope forms a row of images of the star at *RV*, one image for each wave-length. This row of images constitutes the *spectrum* of the star, and it may be examined by the magnifying-glass (eye-piece) *E*.

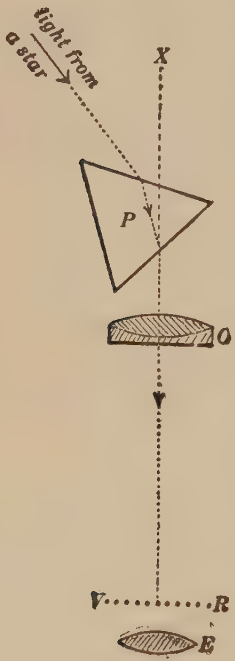

Fig. 148.

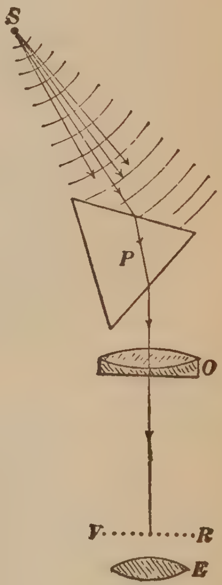

Fig. 149.

If the attempt is made to use the arrangement shown in Fig. 148 for looking at a point-source of light near at hand instead of looking at a star, then the images at *VR* will not be sharply defined, because the light from the near source S will enter the prism P as a series of spherical waves, as shown in Fig. 149; these spherical waves are refracted at the plane surface of the prism *subject to spherical aberration* as explained in Art. 44 and as represented in Fig. 45; and the result is that the waves which emerge from the prism P are non-spherical and they cannot be sharply focussed by the lens O. In order to overcome this difficulty a lens is placed between the point-source S and the prism P in Fig. 149, the point-source S being at the focal point of the lens. With this arrangement, which is shown in Fig. 150, the beam of divergent rays from S is converted into a beam of parallel rays by the lens C (that is, the spherical waves from S are converted into plane waves by the lens) and this beam of parallel rays (plane waves) is refracted by the prism without spherical aberration.

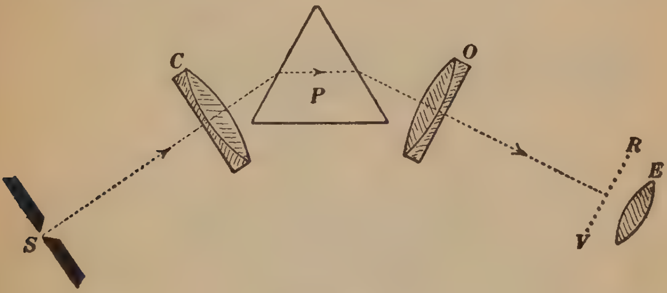

Fig. 150.

The light which is to be analyzed passes through a very narrow slit S, Fig. 150, between two straight metal edges. This slit is then, in effect, the source of the light, and it is at the focal point of an achromatic lens C, which is called the *collimating lens* of the spectroscope. After passing through the collimating lens, the light passes through the prism P and is then focussed by the lens O

as a series of lines (images of the slit) at RV. These images of the slit are examined by the magnifying-glass (eye-piece) E. The band of images at RV is called the *spectrum* and the individual images of the slit are called the *lines of the spectrum.* The slit S and the lens C are mounted at the ends of a short tube which is called the *collimator,* and the lens O and the eye-piece E are mounted at the ends of a tube which is called the *telescope.* The eye lens E is always a compound lens ; usually like Fig. 136.

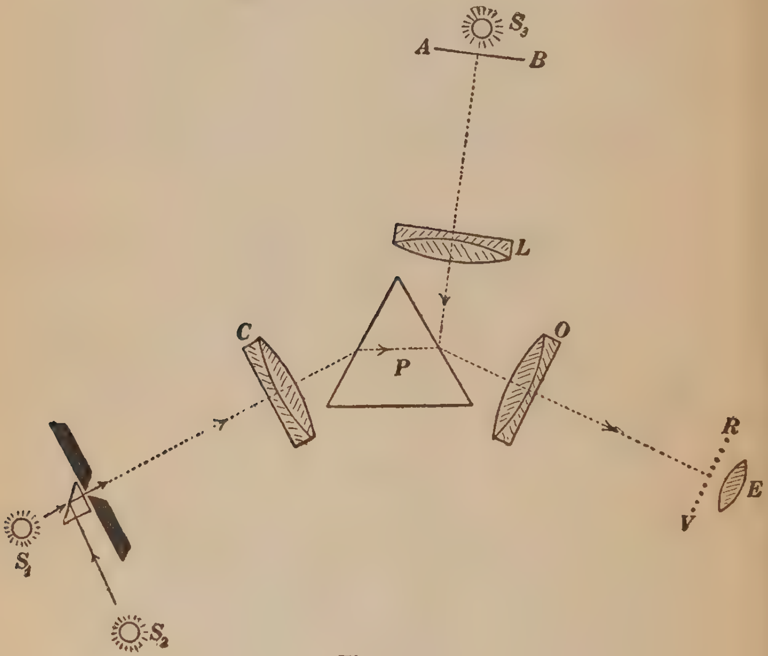

Fig. 151.

The usual arrangement of the spectroscope, as used by chemists, is shown in Fig. 151 in which two independent sources S_1 and S_2 are arranged to send light through the slit by covering one end of the slit with a total reflecting prism so that light from S_1 passes through the upper end of the slit and light from S_2 passes through the lower end of the slit. The two spectra (the spectrum of S_1 and the spectrum of S_2) are then seen one above the other in the spectroscope and they may be compared with great ease. The total reflecting prism is often called the *comparison prism.*

In order to be able to read off the positions of the images of the slit at RV, a lamp S_3 is arranged to send light through a transparent scale AB which is at the focal point of the lens L. The light which passes through the scale AB is reflected from the face of the prism into the telescope lens O, and an image of the scale AB is formed in the plane RV so that the positions of the images of the slit may be read off. A general view of a spectroscope is shown in Fig. 152.

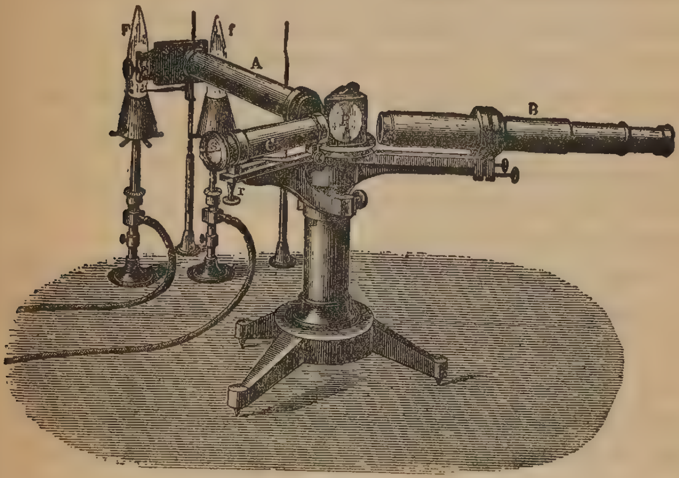

Fig. 152.

83. Continuous spectra. When light from a hot solid or liquid is analyzed by a spectroscope, a continuous band of images of the slit is produced at *RV*, Fig. 150. Such a spectrum is called a *continuous spectrum*. Light which gives a continuous spectrum contains wave-trains of every gradation of wave-length. The candle flame, the petroleum flame, and the gas flame give continuous spectra. The light from such flames is given off by hot particles of solid carbon.

84. Bright-line spectra. When the light which is emitted by a hot vapor or gas is analyzed by a spectroscope, a group of distinctly separate images of the slit is produced at *RV*, Fig. 150. Such a spectrum is called a *bright-line spectrum* inasmuch as the separate images of the slit appear as bright lines. Light which gives a bright-line spectrum contains wave-trains of certain definite wave-lengths only.

Every gas or vapor has a characteristic spectrum, that is to say, a characteristic grouping of images of the slit in a spectroscope.

85. Dark-line spectra. When an intense beam of white light, containing all wave-lengths, is passed through a relatively cool vapor or gas and then analyzed by the spectroscope, dark lines (missing images of the slit) are seen where bright lines would be located in the direct spectrum of the vapor or gas. That is to say, a relatively cool vapor absorbs those particular wave-trains which it would itself give off if hot. This relation between the bright-line spectrum of a vapor or gas and the dark-line spectrum of the vapor or gas was discovered by Bunsen and Kirchoff in 1865. Using the flame of a Bunsen burner and charging it with sodium vapor by the vaporization of common salt, they obtained the ordinary bright-line spectrum of sodium. Then passing an intense beam of white light from a lime-light through the Bunsen flame into the slit, it was found that the absorption of the sodium vapor was such as to leave relatively dark lines in place of the bright lines given by the flame alone.

The most striking dark-line spectrum is the solar spectrum, which shows a great number of dark lines. The dark lines in the solar spectrum were first studied by Fraunhofer in 1819 and they are called *Fraunhofer's lines*. The more prominent of these lines are designated by the letters B, C, D, E, b, F, G, H_1, and H_2, in order, beginning at the red end of the spectrum.

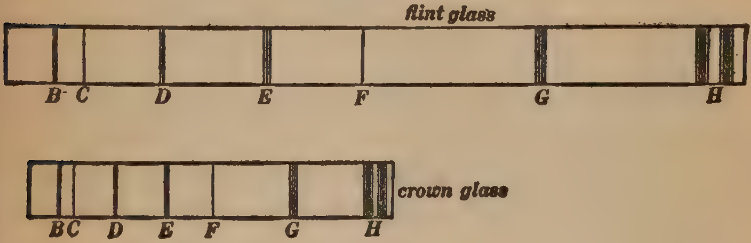

Fig. 153.

Groups of dark lines in the solar spectrum are found to coincide with groups of bright lines given by iron vapor, sodium vapor, hydrogen and other elements, which is an indication that relatively cool vapors of these substances exist in the gases which surround the sun. Certain dark lines in the solar spectrum vary in intensity with the altitude of the sun above the horizon. These lines are due to the absorption of the earth's atmosphere.

86. The spectrometer is a spectroscope which is provided with a divided circle by means of which the position of the axis of the telescope OE in Fig. 150 may be read off, The center of the circle is beneath the prism P, Fig. 150. the telescope OE is carried on an arm which is pivoted at the center of the circle, and cross-wires are stretched across the focal plane RV.

87. The direct-vision spectroscope. Consider a prism of flint glass and a prism of crown glass which give the same deflection for the middle of the spectrum (by deflection is meant the angle α in Fig. 43). Then the flint-glass prism gives a much longer spectrum than the crown-glass prism as shown in Fig. 153; and the two prisms, if arranged as shown in Fig. 154, disperse a beam of white light without deflecting the middle portion of the spectrum.

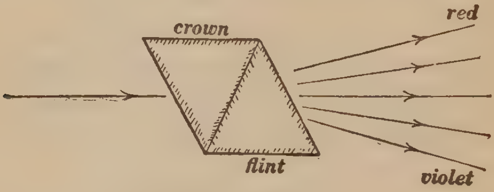

Fig. 154.

The ordinary spectroscope would be an awkward instrument to use if one were to attempt to hold it in the hands and keep the *collimator CS*, Fig. 150, directed towards a source of light while looking into the *telescope OE*. Figure 155 shows the optical parts of a *direct-vision spectroscope*; S is the slit, CL is the collimating lens, CFC is a combination of crown-glass and flint-glass prisms which does not deflect the middle part of the spectrum, but which gives a moderate amount of disper-

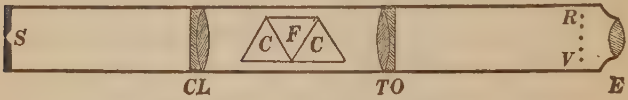

Fig. 155.

sion, *TO* is the telescope objective, *VR* is the spectrum (row of images of the slit), and *E* is the eye lens.

88. The achromatic lens. Consider a prism of flint glass and a prism of crown glass which give spectra of the same length. Then the crown-glass prism gives a much greater deflection (by deflection is meant the angle α in Fig. 43) than the flint-glass prism. Two such prisms arranged as shown in Fig. 154 would deflect a beam of light without perceptibly dispersing it. Such an arrangement might be called an *achromatic prism*. An achromatic lens consists of a converging lens *CC* of crown

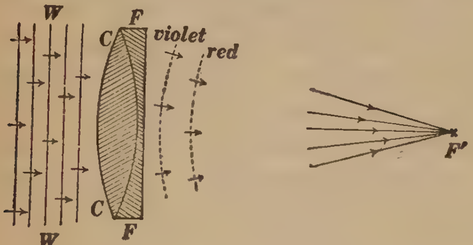

Fig. 156.

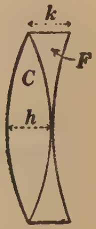

Fig. 157.

glass and a diverging lens *FF* of flint glass arranged as shown in Fig. 156 in which *WW* represents a series of plane waves of white light. After passing through the compound lens, waves of all wave-lengths have approximately the same curvature and are focussed at the point *F'*. Short waves (violet light) travel slower in both kinds of glass than long waves (red light), but the difference between the velocity of red light and the velocity of violet light is much greater in flint glass than in crown glass, and the effect of the double lens of Fig. 156 is to cause the central parts of all the waves (of all wave-lengths) to fall behind the edge portions by the same amount.

The following outline of the theory of the achromatic lens is developed by the method of Art. 52 for the sake of easy intelligibility.

Let C_r and C_b be the indices of refraction of crown glass for red light and for blue light, respectively, and let F_r and F_b be the indices of refraction of flint glass for red light and for blue light, respectively. Consider a converging lens of crown glass C, Fig. 157, and a diverging lens of flint glass F. The crown lens is assumed to come to a sharp edge and the flint lens is assumed to be of zero thickness at the center for the sake of simplicity. Let h be the thickness of the crown lens at its center and k the thickness of the flint lens at its edge. Consider a plane wave of red light passing through the two lenses. The central part of the crown lens is equivalent to a distance C_rh to be traveled in air, and the edge portion of the flint lens is equivalent to a distance F_rk to be traveled in air. Therefore, the central portion of a red wave is retarded with respect to the edge portions by an amount which is equal to $(C_rh - F_rk)$. Consider a plane wave of blue light passing through the two lenses. The central portion of the crown lens is equivalent to a distance C_bh to be traveled in air, and the edge portion of the flint lens is equivalent to a distance F_rk to be traveled in air. Therefore the central portion of the blue wave is retarded with respect to the edge portion by an amount which is equal to $(C_bh - F_bk)$. If both sets of waves, red and blue, are to be focussed at the same point, the retardation of the central portion with respect to the edge portions must be the same for both, that is,

$$C_rh - F_rk = C_bh - F_bk$$

whence

$$\frac{k}{h} = \frac{C_r - C_b}{F_r - F_b}$$

This is the necessary relation between *h* and *k* to give achromatism. The absolute values of *h* and *k* depend upon the diameter of the lens and the desired focal length. The curvatures of the various surfaces in so far as they are not fixed by the absolute values of *h* and *k* and the diameter of the lenses may be chosen so as to eliminate spherical aberration.*

It is impossible to completely compensate the dispersion of a converging crown-glass lens by the dispersion of a diverging flint-glass lens for reasons which are evident from Fig. 158. This figure shows spectra of the same total length (Fraunhofer's *B*-line to Fraunhofer's *H*-lines) formed by flint-glass and crown-glass prisms, and it also shows that the *spectra are not of the same length* as measured between other pairs of lines of the spectrum.

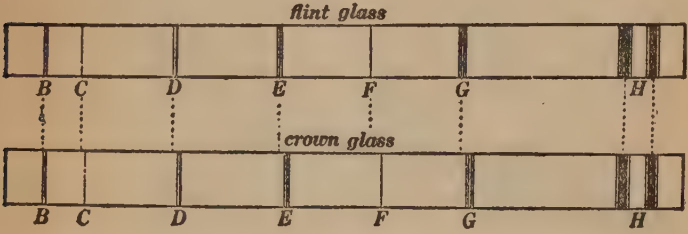

Fig. 158.

89. The spectro photometer is a combination of a spectroscope and a photometer for comparing the intensities of two beams of light wave-length by wave-length.

One of the best forms of spectrophotometer is the spectrophotometer of Lummer and Brodhun, the essential features of which are shown in Fig. 159, in which *L-B* is a Lummer-Brodhun prism-set. The observer's eye, placed at the narrow slit *E*, looks through the lens *TT* and the glass prism *P* and is focussed on the diagonal face *ff* of the Lummer-Brodhun set so that this diagonal face is the observer's field of view. The observer sees the central portion of the field of view illuminated by light of one wave-length from the diffusing plate *G*, whereas the edge portions of the field of view are illuminated by light of the same wave-length from the diffusing plate *F*. The two diffusing plates *F* and *G* are illuminated by the comparison lamp and the test lamp, respectively, and the distances *D* and *D′* are adjusted until the observer's field of view is uniformly illuminated. Then the ratio of brightness of the two lamps for the given wave-length is equal to the ratio of the squares of the distances *D* and *D′*. In some forms of the Lummer-Brodhun spectrophotometer, the observer's field of view is brought to uniform intensity of illumination by adjusting the widths of the two slits *S* and *S′*.

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* An example of the complete calculation of an achromatic, aplanatic telescope objective, according to Gauss, is given by Otto Lummer in Müller-Pouillet's *Lehrbuch der Physik*, Vol. II, Part I, pages 573–579.

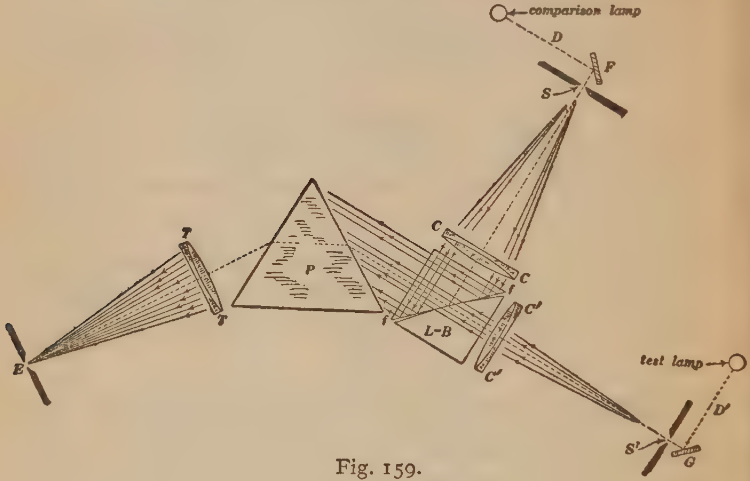

Fig. 159.

Figure 160 shows the results of a spectrophotometric comparison of daylight (curve *DD*) taken as a standard, with the light from a carbon-filament electric lamp (curve *C*), with the light from a tungsten-filament electric lamp (curve *T*), and with the light from the crater of a carbon-arc lamp. The abscissas are wave-lengths in millionths of a meter, the abscissa 0·7 corresponds to the red end of the spectrum, and the abscissa 0·4 corresponds to the violet end of the spectrum. The meaning of these curves

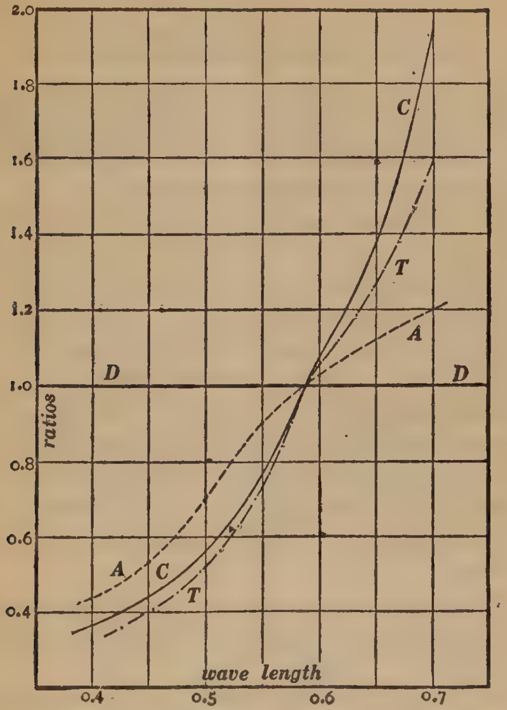

Fig. 160.
CC, carbon-filament lamp; *TT*, tungsten-filament lamp; *AA*, carbon-arc lamp; *DD*, daylight.

is as follows: *For the same intensity at the sodium line* (wave-length $0{\cdot}589 \times 10^{-6}$ meter), the light from a carbon-filament lamp is 1·93 times as intense as daylight in the extreme red and 0·36 as intense as daylight in the extreme violet.

LEADING QUESTIONS

L. Q. 50. A thing which is made up of dissimilar parts is said to be non-homogeneous. Why is ordinary white light said to be non-homogeneous? In what respects are the constituent parts of white light dissimilar?

L. Q. 51. What is meant by mono-chromatic or homogeneous light?

L. Q. 52. Why does not the lens O in Fig. 149 form sharply defined images of S at V. . . . . .R?

L. Q. 53. Why is the lens C in Fig. 150 necessary?

L. Q. 54. Where must the slit S be placed with reference to the collimating lens C in Fig. 150? Why?

L. Q. 55. What does one see in a spectroscope when light of one wave-length passes through the slit? Why?

L. Q. 56. What does one see in a spectroscope when light from a carbon-filament electric lamp passes through the slit? Why?

L. Q. 57. What does one see in a spectroscope when light from a Bunsen flame charged with calcium vapor passes through the slit? Why?

L. Q. 58. What does one see in a spectroscope when sunlight passes through the slit? Why?

L. Q. 59. For the same intensity at the sodium line (wave-length $0{\cdot}589 \times 10^{-6}$ meter) what is the ratio D/T at wave-lengths $0{\cdot}4 \times 10^{-6}$ meter and $0{\cdot}7 \times 10^{-6}$ meter, where D is daylight and T is light from a tungsten-filament electric lamp?

CHAPTER IX

INTERFERENCE AND DIFFRACTION*

90. Similar wave sources. Two sources which send out waves in exactly the same rhythm are called *similar sources.*

Figure 161 shows a bent rod *RR* with its ends touching a water surface, and if the bent rod is moved up and down at a frequency *n*, the ends of the rod will be similar sources of water waves.

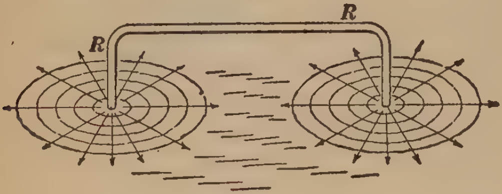

Fig. 161.

Figure 162 shows a small whistle *W* set into one side of a large tube *TT*. When the whistle is blown it sends out sound waves of a definite frequency, these waves spread out in the atmosphere from the open ends *OO′*

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* A very complete discussion of interference is given on pages 139–210 of Preston's *Theory of Light*. Michelson's *Light Waves and their Uses* is devoted wholly to the phenomena of interference and to the uses of the interferometer.

A very good discussion of diffraction is given on pages 211–293 of Preston's *Theory of Light*. See also Mann and Millikan's translation of Drude's *Theory of Optics*, pages 159–241.

The theory of diffraction was first developed by Fresnel and some of Fresnel's original memoirs are of very great interest. See Fresnel's *Complete Works* (in French), Vol. I, pages 1–382.

of the tube, and these open ends of the tube constitute similar sources of sound waves.

Figure 163 shows a piece of cardboard *S* which is illuminated by the beam of light *A* which comes directly from a point source *O*, and also by the beam *B* which is reflected from the mirror and comes therefore from the point *O′* which is the image of *O*. The two points *O* and *O′* are to be thought of as similar sources of light waves.

Consider two similar wave sources *O* and *O′* in Fig. 164, and let λ be the wave-length of the emitted waves.

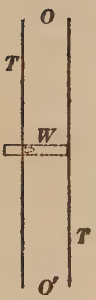

Fig. 162.

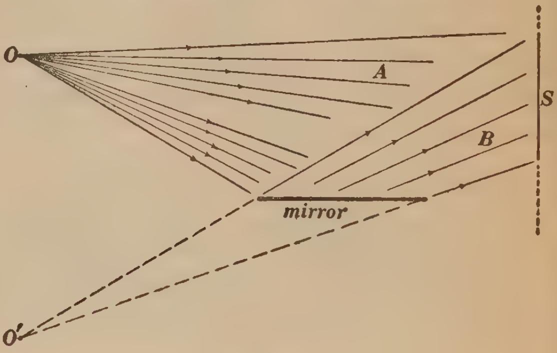

Fig. 163.

Consider any point *q* whose distance from *O* is equal to its distance from *O′* or differs therefrom by any whole number of wave-lengths. The wave-trains from *O* and *O′* are *in phase* with each other at any such point so that the disturbance at *q* is the *sum* of the disturbances which would be produced at *q* by the separate wave-trains.*

Consider any point *p* whose distance from *O* differs from its distance from *O′* by an odd number of half-wave-lengths. The wave-trains from *O* and *O′* are *opposite in*

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* Using the familiar terms relating to water waves, this means that *crests* of both wave-trains reach *q* simultaneously, or that *hollows* of both wave-trains reach *q* simultaneously.

phase[*] at any such point so that the disturbance at p is the *difference* of the disturbances which would be produced at p by the separate wave-trains.

All the points q which satisfy the above condition lie on the dotted hyperbolas[†] in Fig. 164, and all the points p which satisfy the above condition lie on the full-line hyperbolas in Fig. 164.

Everywhere on the dotted lines in Fig. 164 the combined disturbance due to the two sources O and O' is great, and everywhere on the full lines the combined or resultant disturbance is small.

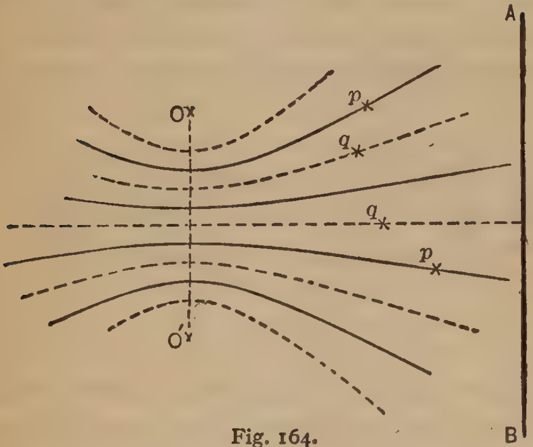

Fig. 164.

Let O and O', Fig. 164, be similar light sources. Then the hyperbolas in Fig. 164 are to be thought of as representing hyperboloids of revolution (about OO' as an axis), and the illumination on a cardboard screen AB will be a series of bright and dark bands, bright bands along the lines where the dotted hyperboloids cut *AB*, and dark bands along the lines where the full-line hyperboloids cut *AB*. These bright and dark bands on *AB* are called *interference fringes.*

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[*] A *crest* of one wave-train and a *hollow* of the other wave-train reach p simultaneously.

[†] The hyperbola is the locus of a point whose distances from two fixed points have a constant difference.

The above discussion refers to the case in which the sources *O* and *O′* emit light of one wave-length only, and in this case bright and dark bands, all of the same color, will be seen on the cardboard screen *AB*.

If the similar sources *O* and *O′* in Fig. 164 give off white light, then a separate set of fringes or bands will be produced for each wave-length and the effect on the screen will be the superposition of all the sets of fringes. There will be a central white band (where bright bands are located for all wave-lengths). Passing out from this central white band in either direction (up or down in Fig. 164), we come first to a region where the violet light, having the shortest wave-length, is absent so that the illumination of the screen in this region will be white *minus* violet, and so on for blue, green, yellow, and red, in order. Thus it is evident that the screen *AB* will show a multiplicity of colored bands or fringes when the similar sources *O* and *O′* give off white light.

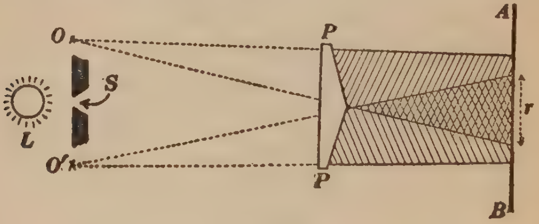

Fig. 165.

The most satisfactory arrangement for showing interference fringes is shown in Fig. 165. This arrangement is due to Fresnel. Light from a lamp *L* passes through a narrow slit *S* and through the double prism *PP*. After passing through *PP* the light virtually comes from two slits *O* and *O′*, and interference bands or fringes are produced on the portion *r* of the cardboard screen *AB*.

Interference of sound. When the sources *O* and *O′* of Fig. 164 emit sound waves of exactly the same frequency, the sound due to *O* and *O′* together is *loud* at any point on a dotted-line-hyperboloid, and *weak* at any point on a full-line-hyperboloid, and if the sources *O* and *O′* give out musical tones which differ very slightly in frequency a very interesting effect is produced. Suppose, for example, that *O* gives out 251 waves per second and that *O′* gives out 250 waves per second. Then the hyperboloidal sheets *qqq* . . . *ppp* will move slowly away from *O* and towards *O′*, while *O* gains one-half of a vibration relative to *O′*, that is to say, during half a second, a given hyperboloidal sheet will move from a *q* to a *p* position, and the sound at any particular point in space will rise and fall in loudness as the hyperboloidal sheets pass by. These pulsations of loudness are called *beats*.

91. The colors of thin plates or films due to interference. Everyone is familiar with the brilliant colors of soap bubbles. A very thin plate or film of any transparent substance reflects light chiefly of certain definite wavelengths, and therefore such a film appears to be brilliantly colored. Before undertaking to discuss this effect it is necessary to consider an important difference in the character of the reflection from front and back faces of a thin plate or film, as follows.

Everyone is familiar with *crests* and *hollows* in a succession (a train) of water waves, and a train of light waves has what we may call "crests" and "hollows."

Let *AA*, Figs. 166 and 167, represent a train of plane light waves, full lines being "crests" and dotted lines being "hollows."

Figure 166 shows a crest being reflected as a crest and a hollow being reflected as a hollow from the front face of the glass plate. This kind of reflection is called *reflection without phase reversal.**

Figure 167 shows a crest being reflected *as a hollow*, and a hollow being reflected *as a crest* from the back face of the glass plate. This kind of reflection is called *reflection with reversal of phase.**

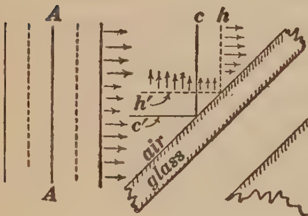

Fig. 166.

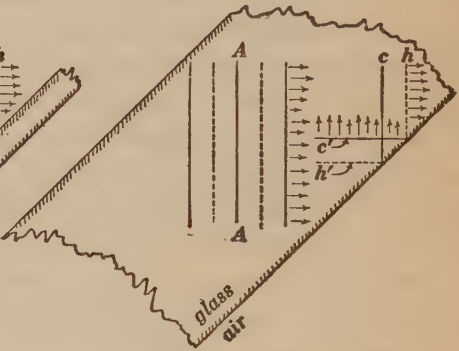

Fig. 167.

Let *PP*, Fig. 168, represent a thin soap film or a thin plate of glass, and let *TT* represent a train of plane light waves of wave-length λ (distance crest to crest, or hollow to hollow). Let us fix our attention on a chosen wave-crest of the train *TT*. This chosen wave-crest is partly reflected as a crest c' from the front face of *PP*; and a remnant of the chosen wave-crest enters *PP*, is reflected *as a hollow* from the back of *PP*, and finally comes out of the front of PP as the hollow h'. *It is desired to find the distance d that h' is behind c'.* While the remnant of the chosen crest (or hollow, as it becomes at b) is traveling the distance $2a$ from f to b to g in glass, the reflected part c' of the chosen crest will travel μ times as far in air, where μ is the index of refraction of the glass. Therefore the distance fc' is equal to $2\mu a$, so that the distance d is $2\mu a - l$, where l is the distance fe.

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* These statements as given are incomplete, but they are correct as used in the following discussion.

Now a whole train of waves is reflected from the front face of PP, and this wave-train is represented by the full

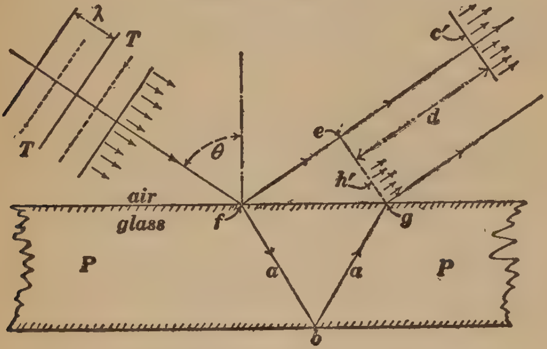

Fig. 168.

and dotted lines $T'T'$ in Figs. 169 and 170. Also a whole train of waves is reflected from the back face of PP, and this wave-train is represented by the full and dotted lines $T''T''$ in Figs. 169 and 170. Of course these two wave-trains $T'T'$ and $T''T''$ overlap each other, but they are shown separately in Figs. 169 and 170.

Looking at Fig. 169 (where $d = 5\lambda/2$), we see that the overlapping wave-trains $T'T'$ and $T''T''$ will correspond crest to crest and hollow to hollow if d equals $n\lambda/2$, where n is *any odd number.* In this case the wave-trains $T'T'$ and $T''T''$ blend and give a single resultant wave-train of great intensity. That is to say, in this case light of the particular wave-length λ is reflected by PP (counting both faces) in large quantity.

Looking at Fig. 170 (where $d = 6\lambda/2$), we see that the overlapping wave-trains $T'T'$ and $T''T''$ will correspond crest to hollow and hollow to crest if d equals $n\lambda/2$, where n is *any even number*. In this case the wave-trains $T'T'$ and $T''T''$ blend and give a single resultant wave-train of low (or even zero) intensity. That is to say, in this case light of the particular wave-length λ is reflected by PP (counting both faces) in small quantity or not at all.

The distance $d = 2\mu a - l$ is called the retardation of h' with respect to c' in Figs. 168, 169, and 170, and it can be easily calculated from the known angle of incidence of the light TT in Fig. 168, the thickness of PP and the value of μ being known.

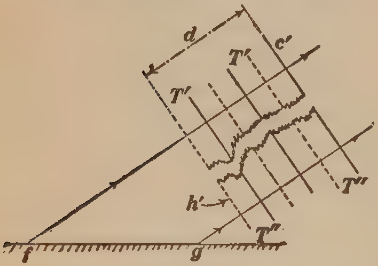

Fig. 169.

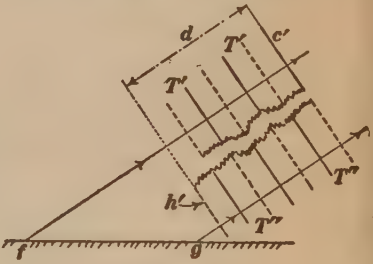

Fig. 170.

The above conditions may be stated in a slightly altered form as follows :

Case I. A large amount of light of wave-length λ will be reflected by PP if $2d$ contains λ an *odd number* of times.

Case II. A small amount of light of wave-length λ will be reflected by PP if $2d$ contains λ an *even number* of times.

If the plate *PP* is less than a few ten-thousandths of a centimeter in thickness, then, of all the values of λ within the limits of the visible spectrum ($\lambda = 75 \times 10^{-6}$ centimeter for red, to $\lambda = 39 \times 10^{-6}$ centimeter for violet) only one or two will be contained an odd number of times in $2d$; therefore such a thin plate reflects, chiefly, light of one or two wave-lengths and appears brilliantly colored. But if the plate is much more than a few ten-thousandths of a centimeter in thickness, then many values of λ within the limits of the visible spectrum will be contained an odd number of times in $2d$; therefore such a plate reflects many wave-lengths throughout the spectrum and these blend together as approximately white light.

92. The Michelson interferometer is an instrument which can be used to measure the wave-length of monochromatic light with extreme precision, or, having light of known wave-length, it can be used to measure short distances with extreme precision.

Figure 171 is a general view of a Michelson interferometer, and Fig. 172 is a diagram of the same; *A* and *B* are two exactly similar glass plates, except that *B* has no silver on it at all, whereas the face *bb* of *A* has a coating of silver so thin that about half of a beam of light which reaches the face *bb* is reflected by the silver and about half passes on through the silver; *C* and *D* are two highly polished metal reflectors.

Let the lines *TT*, Fig. 172, represent the crests of a train of plane waves coming towards *AA*, and let us follow a *particular crest* of this train. Half of this crest is reflected at the back face *bb* as a hollow *h* and it travels towards *D*; and half of the particular crest goes on through *bb* towards *C* as a crest *c*. The reflected hollow *h* travels to *D* whence it is reflected back to *AA*, and part of it goes on through *AA* as the hollow *h′*. Likewise the crest *c* travels to *C* whence it is reflected back to *AA* and about half of it is reflected from *bb* (which is a front face with respect to the light returning from *C*) as the crest *c′*.

The hollow *h* after it is reflected from *bb* goes once through the glass plate *AA* on its way to *D*, and it passes again through *AA* on its return from *D*. On the other hand, the crest *c* goes once through the glass plate *BB* on its way to *C*, and it passes again through *BB* on its return from *C*. Therefore *c′* and *h′* have passed through the same thickness of glass, and if the air path from *A* to *D* is *d*/2 centimeters shorter than the air path from *A* to *C* (doubled-path *d* centimeters shorter) it is evident that *h′* will be *d* centimeters ahead of *c′*.

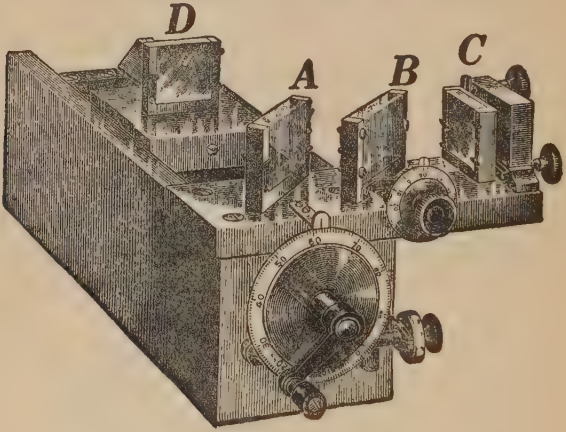

Fig. 171.

Now the hollow *h′* is associated with a complete train of waves *T′T′*, Fig. 173, and the crest *c′* is associated with a complete train of waves *T″T″*. These two wave-trains overlap, but to avoid confusion they are not shown as overlapping in Fig. 173. Therefore, if the retardation d of h' is an odd number of half wave-lengths the trains T' and T'' will be crest-on-crest, and the result will be an intense beam of light at E, whereas, if d is an even number of half wave-lengths the two wave-trains T' and T'' will be crest-on-hollow and the beam at E will be very weak (zero intensity if T' and T'' are of the same intensity). That is to say, an eye at E in Fig. 172 sees bright light if d is an odd number of half-wave-lengths, or no light at all if d is an even number of half-wave-lengths.

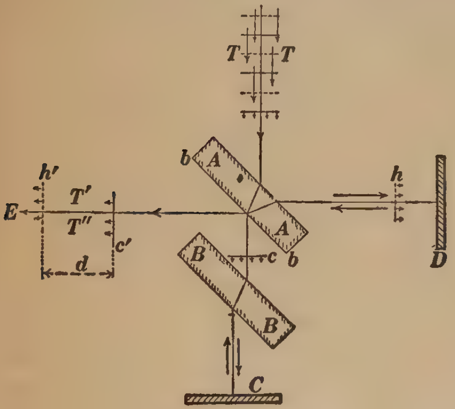

Fig. 172.

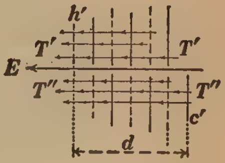

Fig. 173.

The polished silver reflector D is mounted on a sliding carriage and a micrometer screw is arranged to move D towards or away from AA. Let the reflector be moved until the eye at E sees no light, that is, until the field of vision is dark. Then as D is moved the field becomes light and dark alternately, and the distance l that D is moved is equal to $n\lambda/2$, where n is the number of times the field has become dark during the movement and λ is the wave-length of the light TT. Thus if l is measured (by the micrometer screw) and n is counted we can calculate λ, or if λ is known and n is counted we can calculate the value of l.

In the actual use of the instrument the faces of C and D are not exactly at right angles, and the field of vision presents a series of bright and dark bands which move sidewise across the field as D is moved, and $l = n\lambda/2$, where n is the number of bright (or dark) bands which pass by a given point of the field during the movement l.

93. Diffraction. The spreading of a wave disturbance into the region behind an obstacle is called *diffraction.* This spreading action is very prominent in the case of water waves and sound waves, but in the case of light waves it requires special arrangements to make it perceptible. The slight tendency of light waves, as compared with the great tendency of sound waves and water waves to spread into the region behind an obstacle, is due to the very short wave-length of light as compared with the long wave-lengths of water waves and of sound waves.*

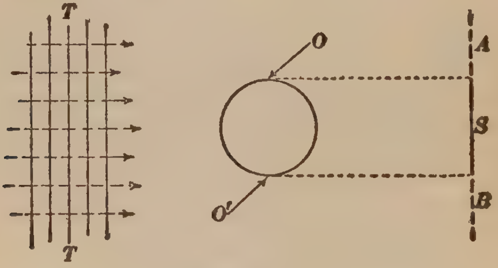

Fig. 174.

Example of diffraction. The circle in Fig. 174 represents a small wire which is perpendicular to the plane of the paper, a beam of parallel rays (plane waves) of monochromatic light TT passes the wire, a shadow S is cast on the cardboard screen AB, and this shadow is filled with many fine bright bands or interference fringes. As a first approximation this effect may be explained as follows: The light which bends around the wire and spreads out in the region between the dotted lines comes mostly from two very small regions O and O', so that O and O' are similar sources, and the illumination of AB by these similar sources gives a series of bright and dark bands as explained in Art. 90.

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* The theory of diffraction was first developed by Fresnel in 1815. See Fresnel's *Œuvres Complètes*, Vol. I, pages 1–382. A very good discussion of this subject is to be found in Edser's *Light for Students*, pages 427–470. Discussions of diffraction are given in Preston's *Theory of Light*, pages 211–393, and in Drude's *Theory of Optics*, translated by Mann and Millikan, pages 159–241.

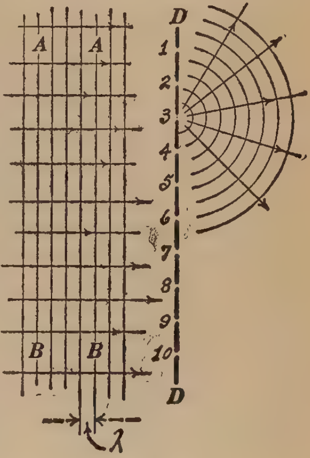

Fig. 175.

94. The diffraction grating. The diffraction grating consists of a series of very fine equidistant straight lines ruled on a glass plate or on a polished metal surface. When the lines are ruled on a glass plate the spaces between the lines are narrow slits through which light can pass, and the following discussion refers explicitly to this type of grating which is called a *transmission grating*. A grating ruled on a polished metal surface is called a *reflection grating*.

The heavy dotted line DD in Figs. 175 and 176 represents a diffraction grating, and the numbered spaces between the dots represent the equidistant slits.

Consider a train of plane light waves AB, Fig. 175, approaching the grating DD. The slits are so extremely narrow that the waves AB pass through the slits and spread out as cylindrical waves or wavelets, as shown by the circles in Fig. 175.

Figure 175 shows the cylindrical wavelets from one slit only, whereas wavelets from many slits are shown in Fig. 176, and these wavelets blend to form a whole series of plane wave-trains, one train of the series being represented by the dotted parallel lines in Fig. 176.

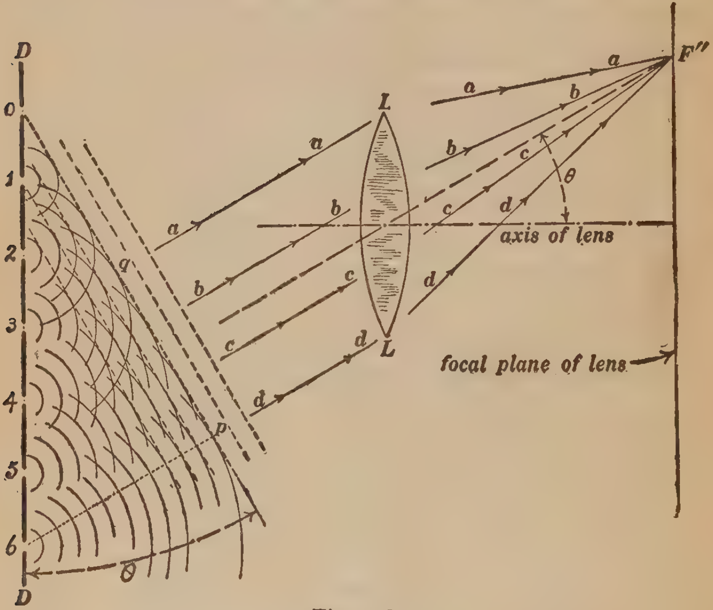

Fig. 176.

The dotted line opq in Fig. 176 is tangent to the nth wavelet from slit No. 1, it is tangent to the $2n$th wavelet from slit No. 2, it is tangent to the $3n$th wavelet from slit No. 4, and so on, where n is any integer. Figure 176 is constructed for $n = 2$. Therefore the distance $p6$ in Fig. 176 is equal to $6n\lambda$, and, of course, the distance $o6$ is equal to $6d$, where λ is the wave-length of the incident light AB in Fig. 175, and d is the distance between slits, center to center. Therefore we have

$$\sin\theta = \frac{n\lambda}{d} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

where θ is the angle shown in Fig. 176 and n is any whole number, positive or negative.

What is the effect of the lens LL on the wave-train which is represented by the dotted lines in Fig. 176? Neglecting lens errors the lens brings this wave-train (or the parallel rays which correspond to the wave-train) to a focus at the point F'' which is determined as the point where the ray through the center of LL cuts the focal plane of LL. Therefore if we think of the wave-train AB in Fig. 175 as having come from a very distant point source S, there will be an image of S at F'' for $n = +2$; or, in general, there will be an image of S for every possible value of n ($n = 0$, $n = \pm 1$, $n = \pm 2$, $n = \pm 3$, etc.), and these images will be located in accordance with equation (i) which determines the value of θ.

The above discussion sets forth the purely geometrical theory of the diffraction grating. The physical theory, very briefly stated, is as follows: Consider the plane pq in Fig. 176. Wavelets from all the slits reach pq simultaneously. Consider the "rays" or "paths" aa, bb, cc, and dd; these paths are all of the same "optical length," meaning that they are all traversed in the same time. Therefore wavelets from all the slits reach the point F'' simultaneously, that is to say, the wavelets are all "in phase" with each other at F'', and therefore the wavelets work together to produce an intense disturbance at F''. At points a little above or below F'' in Fig. 176 the wavelets are "out of phase" with each other and at such points the wavelets annul each other.

The wave train *AB* in Fig. 175 is supposed to come from a very distant point source. In the actual *grating spectroscope* the light to be analyzed passes through a very narrow slit *S* and through a collimating lens *CC* whose focal point is at *S*, thus producing a train of plane waves *AA*, as shown in Fig. 177. The lens *LL* then forms a series of images of the slit *S* at the points F', F'', F''', etc., as explained above, the angles θ', θ'', θ'', etc., being determined by equation (i) for $n = \pm 1$, $n = \pm 2$, $n = \pm 3$, etc.

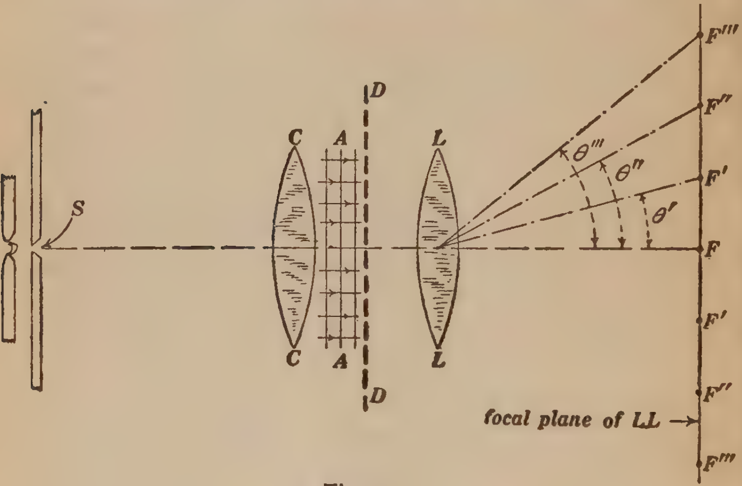

Fig. 177.

All of the above discussion refers to the case in which the incident light *AB* in Figs. 175 and 177 contains light of one wave-length only. In this case there is one image of the slit (one value of θ) at each of the points F', F'', F''', etc., in Fig. 177. But if the incident light *AB* contains a group of wave-lengths, then there will be a *group of corresponding images of the slit* near each of the points F', F'', F''', etc. These groups of images of the slit are called *spectra*; the groups of images of the slit near the two points F' (where $n = \pm 1$) are called *spectra of the first order*, the groups of images of the slit near the two points F'' (where $n = \pm 2$) are called *spectra of the second order*, and so on.

95. Wave-length measurements. The most accurate wave-length measurements are made by means of the Michelson interferometer, but most wave-length data have been determined by means of the grating spectrometer. The grating spectrometer is somewhat like Fig. 177, but arranged so that the angle θ can be measured. Then, if the order n and the grating space d are known, the value of λ can be found from equation (i) of Art. 94.

The accompanying table gives the values of λ corresponding to the principal dark lines (Fraunhofer lines) in the spectrum of sunlight as determined by Louis Bell using a Rowland* grating.

TABLE

Wave-lengths corresponding to Fraunhofer's Lines in the Solar Spectrum.

Line.		Line.	
A .	. 7594·06 × 10^{-8} cm.	F .	. 4861·49 × 10^{-8} cm.
B .	. 6867·46 "	G .	. 4308·07 "
C .	. 6563·06 "		4307·90 "
D_1 .	. 5896·15 "	H_1 .	. 4101·85 "
D_2 .	. 5890·18 "	H_2 .	. 3968·62 "
E_1 .	. 5270·50 "		
E_2 .	. 5269·72 "		

* Diffraction gratings made by Professor H. A. Rowland were for many years (1885–1915) the best that had ever been made.

The concave grating, which does away with the lenses *CC* and *LL* in Fig. 177, is due to Rowland. A simple discussion of the concave grating is given on pages 459–463 of Edser's *Light for Students*. The theory of the concave grating is discussed at length by H. A. Rowland, *American Journal of Science*, Series 3, Vol. XXVI, 1883; by R. T. Glazebrook, *Philosophical Magazine*, Series 5, Vol. XVI, 1883; and by J. S. Ames, *Philosophical Magazine*, Series 5, Vol. XXVII, 1892. A very complete discussion of the diffraction grating is given by C. Runge in Kayser's *Handbuch der Spektroskopie*, Vol. I.

The wave-lengths of the most penetrating X-rays are about 1/10,000 as great as the wave-lengths given in this table.

Wave-length units. The *micron* is one-millionth of a meter or one ten-thousandth of a centimeter. The *Ångstrom unit* is one one-hundred-millionth of a centimeter (1×10^{-8} cm.). Wave-lengths are sometimes expressed in microns, but more frequenly in Ångstrom units.

96. Use of a crystal plate as a diffraction grating for X-rays. The atoms in a crystal, in a crystal of common salt, for example, are arranged uniformly in layers and rows, and for light of exceedingly short wave-length (X-rays) the regularly spaced atoms in a crystal cause a crystal plate to act as a reflection grating. The action is somewhat more complicated than the action which is described in Art. 94 because the grating elements are points in space instead of lines on a surface, but the added complications are wholly geometrical.*

LEADING QUESTIONS

L. Q. 60. What is meant by similar wave sources? Describe a device for producing similar sources of water waves. Describe a device for producing similar sources of sound waves. Is it possible to produce two independent but similar sources of light waves? Describe two devices for producing similar sources of light waves.

L. Q. 61. Light from a tungsten-filament electric lamp is reflected into the slit of a spectroscope by a moderately thick plate of mica, 0·001 cm. thick, for example. Describe the appearance of the spectrum. Explain why the mica plate does not appear colored like a thin soap film.

L. Q. 62. Why is the shadow of a small wire cast by the light of a distant open-arc lamp filled with fine bright lines?

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* A very interesting book is *X-rays and Crystal Structure*, by W. H. and W. L. Bragg, London, 1915.

L. Q. 63. If the light through S in Fig. 177 is white light the central image of the slit at F is white. Explain.

L. Q. 64. Define the micron. Define the Ångstrom unit. State the wave-lengths of Fraunhofer's A and F lines in microns and in Ångstrom units.

PROBLEMS

67a. A glass plate 4×10^{-5} cm. thick is illuminated by a beam of parallel rays. The beam is normal to the plate (angle θ in Fig. 168 equal to 90°). The index of refraction of the glass is 1·50. What wave-lengths within the limits of the visible spectrum ($\lambda = 40 \times 10^{-6}$ cm. to $\lambda = 75 \times 10^{-6}$ cm.) will be intensified in the reflected beam?

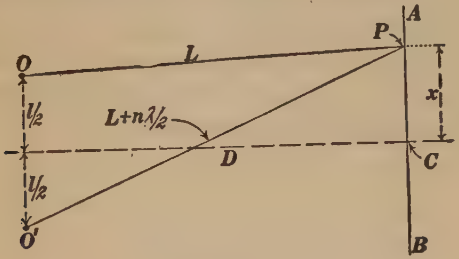

Fig. 178.

67b. Solve the previous problem for the case in which the angle θ is equal to 30°. Ans. $\lambda = 66 \times 10^{-6}$ cm. and $\lambda = 39{\cdot}6 \times 10^{-6}$ cm.

68. Figure 178 is a reproduction of Fig. 164 showing the distance x of a point P from the center of the screen when $O'P$ exceeds OP by the amount $n\lambda/2$ where n is an whole number which is even if the light intensity at P is a maximum and odd if the intensity at P is a minimum. From this figure we have

$$L^2 = D^2 + \left(x - \frac{l}{2}\right)^2$$

and

$$\left(L + \frac{n\lambda}{2}\right)^2 = D^2 + \left(x + \frac{l}{2}\right)^2$$

Expand the binomial terms, drop terms containing l^2 and $n^2\lambda^2$ which are negligible, then put D for L, and derive an expression for x in terms of n, λ, D and l.

69. From the result of the previous problem calculate the number and spacing of the bright lines inside of the geometrical shadow S in Fig. 174 when the diameter of the wire is 0·2 cm., the distance D from the wire to the screen AB is 20 cm., and when λ is 60×10^{-6} cm.

70. The distance apart, center to center, of the slits in the grating DD in Fig. 177 is 0·0003 centimeter. The light AB is white light and the focal length of the lens LL is 50 centimeters. Calculate the distances from the point F to the two points F', and from the point F to the two points F'', and so on, for violet light ($\lambda = 40 \times 10^{-6}$ centimeter) and for red light ($\lambda = 75 \times 10^{-6}$ centimeter). Ans. Distances F to F', for violet light 6·738 centimeters, for red light 12·91 centimeters; F to F'', for violet light 13·83 centimeters, for red light 28·87 centimeters.

CHAPTER X

POLARIZATION AND DOUBLE REFRACTION

97. Polarization of transverse waves. When one end of a stretched rope is held in the hand and moved rapidly up and down or to and fro sidewise a train of waves is sent out along the rope from the hand. These waves are called *transverse waves** because, as the waves pass by, a point on the rope moves to and fro in a direction at right angles to the direction of travel of the waves (the direction of the rope).

If the hand is moved up and down (in a vertical plane), then each particle of the rope will move up and down (in a vertical plane) as the waves travel along, and we have what is called a *plane polarized* train of waves.

If the hand is moved in a small circle around the stretched rope as an axis, then each particle of the rope will travel in a similar circular path as the waves travel along, and we will have what is called a *circularly polarized* train of waves.

If the hand is moved in a small ellipse in a plane at right angles to the rope, then each particle of the rope will move in a similar ellipse as the waves travel along, and we will have what is called an *elliptically polarized* train of waves.

If the hand is moved sidewise and up and down in an irregular way, then an irregular series of waves will travel along the rope.

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* In sound waves the air moves to and fro in the direction in which the waves travel, such waves are called *longitudinal waves.* Transverse waves, only, can be polarized.

98. Polarizer and Analyzer. Case I. Figure 179 shows a stretched rope *RMS* passing through two narrow slits in two parallel boards. The end *R* of the rope is held in the hand, and we will assume that the rope can move to and fro sidewise in the slits without friction.

(*a*) If the hand be moved irregularly up and down and sidewise in rapid succession the waves which pass over the portion *RR* of the rope will be similar in character to the waves in a beam of common (unpolarized) light.

(*b*) The rope can move only up and down in the slit *P*, and therefore the only wave oscillations which pass through *P* will be up and down oscillations, and the waves which pass over the portion *MM* of the rope will be *plane polarized.* The slit *P* is therefore called a *polarizer.*

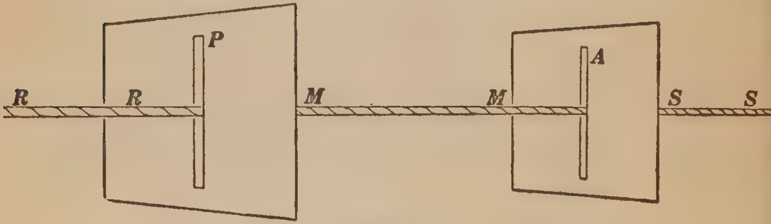

Fig. 179.

(*c*) The slit *A* in Fig. 179 is called an *analyzer*, and its action is as follows: (1) If *A* is parallel to *P*, as shown in the figure, then the plane polarized waves on *MM* pass freely through *A*. (2) If slit *A* is at right angles to *P*, then the plane polarized waves on *MM* cannot pass through at all. (3) If slit *A* is inclined to *P* (the boards are understood to be parallel to each other as stated above), then a component, only, of the wave oscillations on *MM* can pass through A, namely, the component which is parallel to A.

Case II. Figure 180 shows a beam of light RMS passing through two thin plates of tourmaline, the plates being parallel to each other.

(*a*) The light RR is ordinary unpolarized light, the oscillations of which are geometrically similar to the wave oscillations on the part RR of the rope in Fig. 179 as described above.

(*b*) The tourmaline plate PP absorbs* all, or nearly all, of the wave oscillations which are at right angles to the dotted line PP in Fig. 180, and transmits only wave oscillations which are parallel to the dotted line PP This property of tourmaline is called *dichroism*. Many crystalline substances are more or less dichroic. Therefore the beam of light MM in Fig. 180 is a plane polarized beam with its wave oscillations lying in the plane of the paper in the figure, and the tourmaline plate PP is called a *polarizer*.

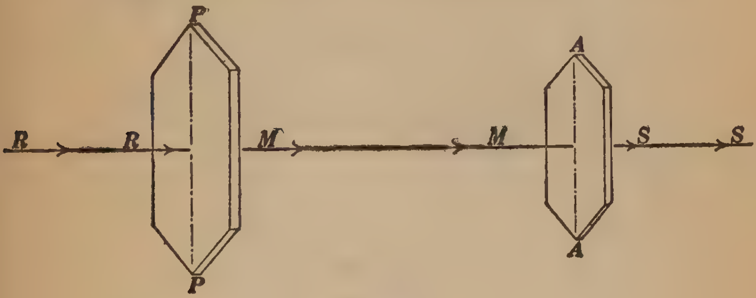

Fig. 180.

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* A mechanical example of this kind of absorption is as follows: Imagine waves traveling along a very thin stretched ribbon, and imagine the ribbon to be in water. Then the edgewise wave oscillations would travel along the ribbon with but little loss by friction, whereas flatwise wave oscillations would be destroyed very quickly by friction.

(*c*) The plate *A* in Fig. 180 is called an *analyzer*, and its action is as follows: (1) If the dotted line *AA* is parallel to *PP*, as shown in the figure, then the plane polarized waves in the beam *MM* pass freely through the plate *AA*. (2) If the dotted line *AA* is at right angles to the dotted line *PP*, then the plane polarized waves in the beam *MM* do not pass through *AA* at all. (3) If the dotted line *AA* is inclined to the dotted line *PP* (the crystal plates *P* and *A* are understood to be parallel to each other as stated above), then a component, only, of the wave oscillations of the beam *MM* can pass through *AA*, namely, the component which is parallel to the dotted line *AA*.

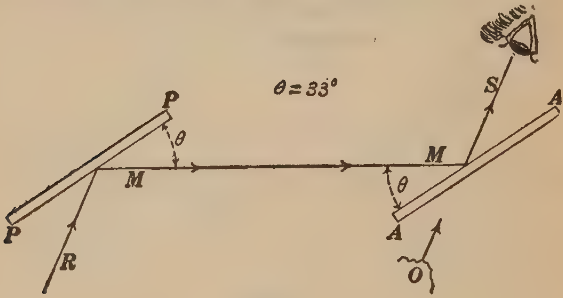


Fig. 181.

Case III. Figure 181 shows a beam of light *RMS* reflected from two clean glass plates *PP* and *AA*, these glass plates being parallel to each other as shown and the angle θ being about 33°.

(*a*) The light *R* is ordinary unpolarized light.

(*b*) The glass plate *PP* reflects only those wave oscillations which are parallel to its surface or at right angles to the plane of the paper in Fig. 181. Therefore the beam *MM* is plane polarized and its wave oscillations are at right angles to the plane of the paper. The glass plate *PP* is therefore a *polarizer*.

(*c*) The glass plate *AA* is called an *analyzer*, and its action is as follows: (1) When *AA* is in the position shown in Fig. 181 it reflects most of the beam *MM* to the observer's eye. (2) If the plate *AA* is turned through ± 90° about *MM* as an axis (without altering the value of θ), then the beam *MM* is not reflected by *AA* at all. (3) If the plate *AA* is in any position between (1) and (2), a component, only, of the wave oscillations in the beam *MM* will be reflected to the observer's eye.

Note 1. The glass plates *PP* and *AA* in Fig. 181 should be made of opaque or black glass for the following reason:

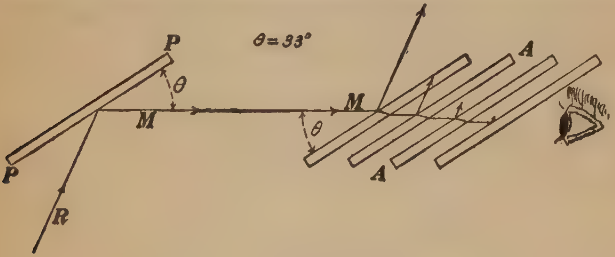


Fig. 182.

If *AA* is made of clear glass some light will reach the observer's eye from objects at *O*, and if *PP* is made of clear glass the beam *MM* will consist partly of light which has come through *PP* from objects behind *PP*.

Note 2. The observer's eye must, of course, be moved as the plate *AA* in Fig. 181 is turned about *MM* as an axis; and this awkward movement of the observer's eye may be obviated by using a pile of glass plates as an analyzer in Fig. 182. Most of the polarized beam is reflected from the first plate of the analyzer in Fig. 182, most of the residue is reflected from the second plate, etc., so that no appreciable quantity of light reaches the observer's eye in Fig. 182; but if the analyzer *AA* in

Fig. 182 is turned ± 90° about *MM* as an axis (without altering the value of θ), then none of the polarized light *MM* is reflected sidewise, but practically the whole of it is transmitted by *AA*.

Case IV. Figure 183 shows a beam of light passing through two Nicol prisms *P* and *A*. The construction and action of the Nicol prism is explained in Art. 103. For present purposes, however, it is sufficient to know that a Nicol prism transmits, only, wave oscillations which are parallel to the shorter diagonal of either of its end faces as shown by the dotted line in Fig. 184.

(*a*) The light *R* in Fig. 183 is ordinary unpolarized light.

(*b*) The Nicol prism *P*, which is called the *polarizer*, transmits, only, wave oscillations which are parallel to the plane of the paper in Fig. 183, so that the beam *MM* is plane polarized.

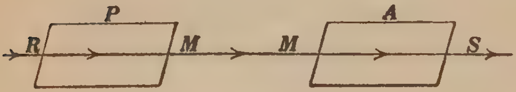
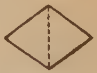


Fig. 183.



end view of A or

Fig. 184.

(*c*) The Nicol prism *A* is called the *analyzer*. (1) When the shorter diagonals of the end faces of *P* and *A* are parallel (as they are in Fig. 183), then nearly the whole of the polarized beam *MM* passes through *A*. (2) When the above-mentioned shorter diagonals are at right angles to each other, then the beam *MM* is not transmitted by *A* at all. (3) When the above-mentioned shorter diagonals are inclined to each other, then a component only of the wave oscillations in the beam *MM* will be transmitted by *A*, namely, the component which is parallel to the shorter diagonal of the end face of *A*.

99. The polariscope. The arrangement shown in either of the figures, 180, 181, 182, and 183, is called a *polariscope*. The position of the analyzer for which no light reaches the observer's eye is called the "crossed position," and the position of the analyzer for which a maximum of light reaches the observer's eye is called the "parallel position."

Many substances, especially crystalline substances, have interesting effects on polarized light and the polariscope is used to observe these effects. The analyzer is usually set in either the "crossed position" or the "parallel position," and the substance to be examined is placed in the path of the beam *MM* between the polarizer and the analyzer. The use of the polariscope is discussed at some length in Arts. 105–108.

100. Polarization of light by reflection. There is always a certain angle θ (see Fig. 181) for which the beam *MM* which is reflected from a clean polished surface *PP* is completely polarized (containing, only, wave oscillations at right angles to the plane of the paper in Fig. 181), *except in the case of metallic surfaces*; and the value of θ for which the reflected beam is completely polarized is called the *polarizing angle* (usually the complement of θ, namely, $90 - \theta$ is called the polarizing angle). The tangent of $90 - \theta$ is equal to the index of refraction of the reflecting substance.*

101. Double refraction. The theory of refraction which is set forth in Chapter IV is limited to what are called isotropic substances (see footnote to Art. 13 of Chapter II). An isotropic substance is a substance like water or annealed glass which has the same physical properties in all directions, whereas a substance like wood or a substance like a crystal of mica or Iceland spar which has different physical properties in different directions is said to be *ælotropic*. Also the theory of refraction* which is set forth in Chapter IV refers only to the direction of the refracted rays.

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* This relation is known as Brewster's law. The complete physical theory of the reflection and refraction of light covers this matter of polarization by reflection and a great deal besides. To give even a bare outline of the fundamental ideas which are involved in the complete theory of reflection is beyond the scope of this text. This matter is discussed in a very satisfactory manner in Drude's *Theory of Optics*, translated by Mann and Millikan, pages 259–302.

Many crystalline substances divide a beam of homogeneous light (light of one wave-length) into two beams by refraction. This phenomenon is called *double refraction*. The crystalline mineral calcite or Iceland spar separates the two beams widely and therefore shows the effect very distinctly.

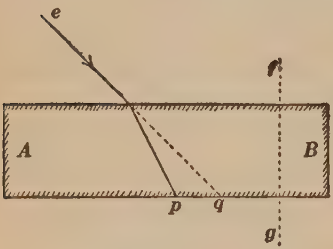

Fig. 185.
Glass plate.

In order to get a clear idea of double refraction, let us consider a ray of light *e*, Fig. 185, which falls upon a glass plate *AB* and is refracted to the point *p*. If the point *p* were a luminous point, then *pe* would be a ray passing out from *p*, and an eye placed at *e* would see the point *p* at *q*. Under these conditions, the plate *AB* may be turned about the line *fg* as an axis without producing any apparent motion of the point *q*, the point *p* being stationary.

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* The complete theory of refraction has to do not only with the direction of the refracted rays, but also with the intensity of these rays and their character as to the direction of their oscillations and their wave-lengths. This part of the theory of refraction is closely associated with the complete theory of reflection. See Drude's *Theory of Optics*, translated by Mann and Millikan, pages 259–302 and pages 308–357.

A beam of common light C, Fig. 186, falling upon a plate AB of Iceland spar, becomes two beams O and X in the crystal. Conversely, a luminous point p, Fig. 186, sends out two particular rays o and x, parallel respectively to O and X, which become parallel rays r and r' in the air, so that an eye placed at e would see *two* images of the point p at q and q', respectively. If the plate of spar AB be turned about the line fg as an axis, while the point p remains stationary, then one of the images q remains stationary as if AB were a plate of glass, and the other image q' moves round q in a small circular path. The refracted ray o, or O, in the spar which corresponds to the stationary image q is called the *ordinary ray* inasmuch as it is refracted in the ordinary way as in glass; and the ray x, or X, in the crystal which corresponds to the moving image q' is called the *extraordinary ray* inasmuch as it is not refracted in the ordinary way as in glass. Some crystals (bi-axial crystals) divide a beam of common light into two beams, neither of which follows the ordinary laws of refraction.

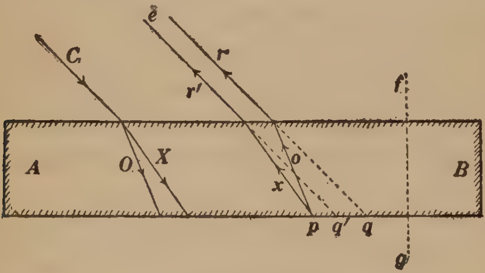

Fig. 186.
Plate of Iceland spar.

The rays r and r', Fig. 186, are completely polarized, and their planes of oscillation are at right angles. This may be shown by holding a tourmaline plate (or a Nicol prism) before the eye at *e*. As the tourmaline plate is turned, one after the other of the images *q* and *q′* becomes invisible.

Axis of symmetry of Iceland spar. Optic axis. Any body may be turned one whole revolution about any axis and be in exactly its initial position so that to the eye it is just the same as if the body had not been turned at all. Consider, on the other hand, a symmetrical body like a cube; there are certain axes about which a cube may be turned through one quarter, two quarters, or three quarters of a revolution and yet appear to the eye to be in exactly the same position as at first. This is an example of the kind of symmetry which is exhibited by crystals, and such an axis is called an axis of symmetry. A crystal of Iceland spar has one axis of symmetry about which it can be turned one-third, or two-thirds, or three-thirds of a revolution and appear as if it had not been turned at all.* Not only is the form of a crystal of Iceland spar symmetrical with respect to this axis, but the physical properties of the substance are also symmetrical with respect thereto. Thus, for example, the heat conductivity of Iceland spar is different in different directions, but it is the same in all directions which are equally inclined to the axis of symmetry of the crystal. This axis of symmetry of the crystal is sometimes called the *optic axis* of the crystal.

102. Application of Huygens' principle to double refraction in Iceland spar. The phenomena of double refraction in Iceland spar were fully analyzed by Huygens, the discoverer of polarized light. Huygens assumed *two* secondary wavelets to enter a plate of Iceland spar from each point on its surface when an incident wave reaches that point; one of these wavelets being a sphere and the other an ellipsoid of revolution. The envelope of the spherical wavelets determines the *ordinary* refracted wave as explained in Art. 34 of Chapter IV, and the envelope of the ellipsoidal wavelets determines the *extraordinary* refracted wave.

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* The student should have access to a paste-board model of a crystal of Iceland spar in order to be able to understand this statement.

The spherical and ellipsoidal wavelets which enter a plate of Iceland spar at a point on its surface when an incident wave reaches that point, are most easily described by imagining a center of disturbance inside of the spar so that the wavelets may be a complete sphere, and a complete ellipsoid, respectively. Let p, Fig. 187, be a center of disturbance in a piece of Iceland spar, and let AB be the axis of symmetry of the crystal (optic axis). The circle represents the spherical wave which passes out from p, and the ellipse represents the ellipsoidal wave. The complete ellipsoid is generated by the rotation of the ellipse about AB as an axis, and the spherical and ellipsoidal wave surfaces touch each other at the poles A and B.

Straight lines drawn outwards from p in Fig. 187 are called *rays*. The disturbance which constitutes the spherical wave and the disturbance which constitutes the ellipsoidal wave may both be thought of as traveling outwards along these rays. The spherical wave is at each point perpendicular to the rays along which it is traveling; but the ellipsoidal wave is not at right angles to the rays at every point.

The velocity of the spherical wave in Iceland spar is the same in all directions, namely, $1/1{\cdot}658$ as great as the velocity of light in air. The velocity of the ellipsoidal wave in the direction of the axis AB is the same as the velocity of the spherical wave, and in all directions at right angles to AB the velocity of the ellipsoidal wave is 1/1·486 as great as the velocity of light in air. This is usually expressed by saying that the *ordinary index of refraction* of Iceland spar is 1·658, and that the *extraordinary index of refraction* of Iceland spar is 1·486. The axis AB in Fig. 187, or any line parallel to AB, is the optic axis of the crystal. Any plane which includes the optic axis is called a *principal plane.*

The oscillations of the spherical wave are everywhere *perpendicular to* the principal planes, that is to say, the movement of a spherical shell of the ether at the instant that the outwardly moving spherical wave reaches it may be imagined as a momentary rotatory twitch of the entire shell about AB as an axis, followed by a reverse twitch which brings the shell into its initial position.

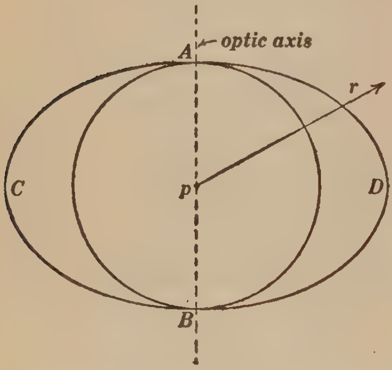

Fig. 187.
Wave surfaces in Iceland spar.*

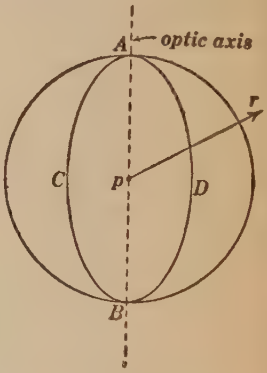

Fig. 188.
Wave surfaces in quartz.*

The oscillations of the ellipsoidal wave are everywhere *in* the principal planes, that is, the movement of an ellipsoidal shell of the ether at the instant that the outwardly

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[*] Iceland spar is called a *negative* crystal and quartz is called a *positive* crystal in view of the difference shown in Figs. 187 and 188.

moving ellipsoidal wave reaches it may be imagined as a momentary twitch which carries each part of the shell a short distance towards one pole of the ellipsoid along a meridian line and then back again. The points *A* and *B* are the poles of the ellipsoid in Fig. 187, and the meridian lines are the lines of intersection of the principal planes with the ellipsoid.

In Iceland spar, the diameter *CD* of the ellipsoidal wave is greater than its diameter *AB* parallel to the optic axis, as shown in Fig. 187. In quartz, however, the diameter *CD* of the ellipsoidal wave is less than its diameter parallel to the optic axis *AB*, as shown in Fig. 188.

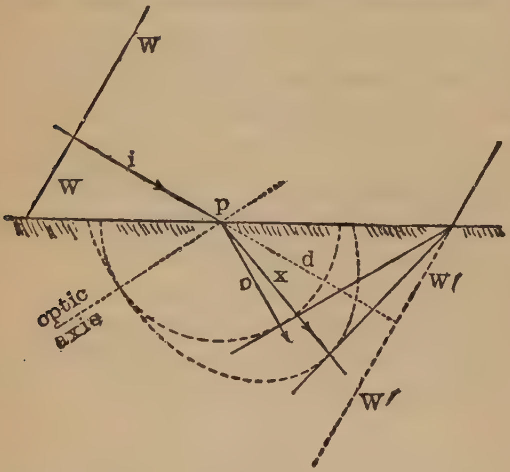

Fig. 189.

Figure 189 shows Huygens' construction for the two refracted waves in Iceland spar. The figure represents the simple case in which the optic axis of the crystal lies in the plane which contains the incident ray *i* and which is perpendicular to the refracting surface, the *plane of* *incidence* as it is called.* *WW* represents an advancing wave in air, and *W'W'* is the position this wave would have reached at *a given instant* if it had not encountered the piece of crystal. It is required to find the positions of the two refracted waves at the given instant. When the wave *WW* reaches the point *p*, that point is a center of disturbance, and it sends two wavelets into the crystal, a spherical wavelet and an ellipsoidal wavelet; and at the given instant the wavelets from *p* have had time to travel the distance *d* in air. Therefore the spherical wavelet in the crystal has a radius equal to $d/1{\cdot}658$, and the ellipsoidal wavelet has a minor axis equal to the radius of the spherical wavelet and a major axis equal to $d/1{\cdot}486$. The direction of the optic axis being given, both wavelets may be drawn as shown. Wavelets from other points in the refracting surface may be determined in a similar manner. The envelope of the spherical wavelets is the ordinary refracted wave, and the envelope of the ellipsoidal wavelets is the extraordinary refracted wave. The ordinary refracted ray *o* is the line drawn from *p* to the point where the spherical wavelet from *p* touches its envelope (ordinary refracted wave). This ray is at right angles to the ordinary refracted wave. The extraordinary refracted ray *x* is the line drawn from *p* to the point where the ellipsoidal wavelet from *p* touches its envelope (extraordinary refracted wave). This ray is generally not at right angles to the extraordinary refracted wave. The direction of oscillation of the ordinary ray *o* is determined by the direction of oscillation of the spherical wavelets at the points where they touch their envelope; this direction is perpendicular to the plane of the paper in Fig. 189. The direction of oscillation of the extraordinary ray x is determined by the direction of oscillation of the ellipsoidal wavelets at the points where they touch their envelope ; this direction is *in* the plane of the paper in Fig. 189. The directions of oscillation of the ordinary and extraordinary rays are most easily specified as follows : Imagine a principal plane (a plane containing the optic axis) drawn so as to include the ray o or x in Fig. 189.

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* The ordinary refracted ray *o* always lies in the plane of incidence. In the case which is represented in Fig. 189, the extraordinary refracted ray *x* also lies in the plane of incidence, but this is not generally the case.

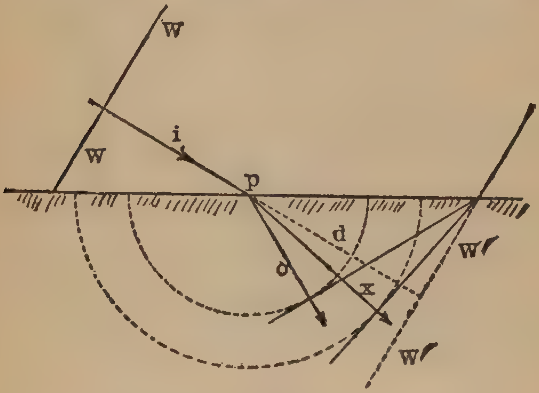


Fig. 190.
Optic axis perpendicular to plane of paper.

The oscillations of the ordinary ray are at right angles to this principal plane, and the oscillations of the extraordinary ray are *in* this principal plane.

Figure 190 shows Huygens' construction for the case in which the optic axis is parallel to the refracting surface and at right angles to the incident ray i, or, in other words, the optic axis is perpendicular to the plane of incidence (the plane of the paper in the figure). The significance of the two dotted circles in Fig. 190 may be understood by referring to Fig. 187. These circles are the equatorial sections of the sphere and ellipsoid, respectively. The direction of oscillation of the ordinary ray is *in* the plane of the paper and the direction of oscillation of the extraordinary ray is at right angles to the plane of the paper in Fig. 190.

Figure 191 shows Huygens' construction for an incident wave *WW* which is parallel to the refracting surface, the direction of the optic axis being indicated by the short dotted line. Here the ordinary and extraordinary rays coincide, but there are two distinct refracted waves nevertheless.

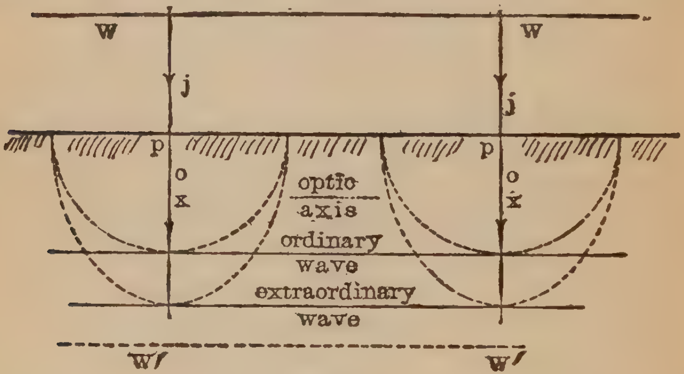
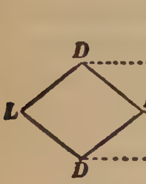


Fig. 191.

103. The Nicol prism. A beam of completely polarized light (plane polarized) can be obtained by reflection from a clean glass plate as explained in Art. 98. A more convenient arrangement, however, for obtaining a beam of completely polarized light is the Nicol prism which is constructed as follows: A crystal of Iceland spar is reduced to the form shown in Fig. 192, by splitting off layers of the crystal.* The line *aa* in Fig. 192 shows the

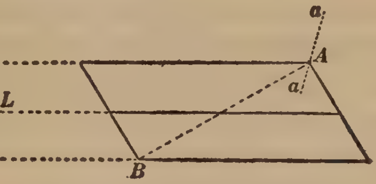
* A crystal of Iceland spar has three cleavage planes, and the faces of the rhomb in Fig. 192 are cleavage planes of the crystal.

direction of the optic axis. This rhomb of spar is sawed along the dotted line AB perpendicularly to the plane of the paper in Fig. 192*b*. The sawed faces and end faces of the rhomb are polished, and the two blocks are cemented together again in their original positions with a thin layer of Canada balsam between them.



end view

Fig. 192*a*.



side view

Fig. 192*b*.

A beam of common light C, Fig. 193, is broken up by the prism into two beams o and x. The ordinary beam o is totally reflected to one side by the layer of balsam, while the extraordinary beam passes on through the prism as shown. Therefore, the light R which emerges from the prism is completely polarized, and its direction of oscillation is *in* the plane of the paper in Fig. 193 (parallel to the shorter diagonal DD of the end face of the Nicol prism).

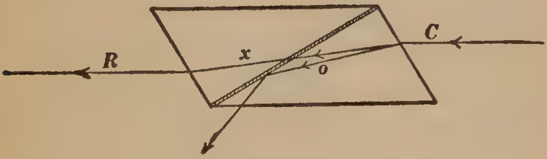


Fig. 193.

The reason for the total reflection of the ordinary ray is as follows: The index of refraction of the spar for the ordinary ray is 1·658 and the index of refraction of the spar for the extraordinary ray is between 1·486 and 1·658, according to its direction, or, say, 1·550. The index of refraction of the balsam is between 1·550 and 1·658. That is, the extraordinary ray goes from a "rare" to a "dense" medium in going from spar into balsam, and the ordinary ray would go from a dense to a rare medium in going from spar into balsam, the words dense and rare referring to large or small index of refraction, respectively. Therefore, the ordinary ray may be totally reflected if it strikes the balsam layer at sufficiently oblique incidence, as explained in Art. 42 of Chapter IV. On the other hand, the extraordinary ray can only be partially reflected however oblique its incidence.*

104. The action of the Nicol prism as analyzer. When a beam of common (unpolarized) light enters a Nicol prism, the beam is converted into two polarized beams o and x, as shown in Fig. 193; and the beam x passes through the prism whereas the beam o is totally reflected to one side as explained above. When the Nicol prism is used as analyzer, however, as shown in Fig. 183, then the effect is as follows: (1) If the wave oscillations of the incident beam MM are parallel to the shorter diagonal DD of the end face, then the incident beam becomes wholly the *extraordinary* beam x in Fig. 193 (there is then no ordinary beam o), and this beam x passes through the prism. (2) If the wave oscillations of MM are at right angles to DD, then the incident beam becomes wholly the *ordinary* beam o in Fig. 193 (there is then no extraordinary beam x), and this beam o is totally reflected to one side. (3) Let us suppose that the vector a in Fig. 194

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* Another device which is sometimes used for completely separating the rays o and x in Iceland spar and thereby producing a beam of plane polarized light, is Rochon's prism. See Preston's *Theory of Light*, page 320.

represents in magnitude and direction the wave oscillations of the incident beam *MM*. The beam *MM* is then resolved by the crystalline material of the prism into two beams, ordinary and extraordinary, of which the amplitudes of oscillation are represented by the lines *o* and *x*, respectively. The extraordinary ray (which passes through the prism) has an amplitude *x* which is equal to $a \cos \phi$, and the intensity *T* of the transmitted beam is to the intensity *I* of the incident beam as $a^2 \cos^2 \phi$ is to a^2, because the intensity of a beam of light is proportional to the square of its amplitude of oscillation. Therefore,

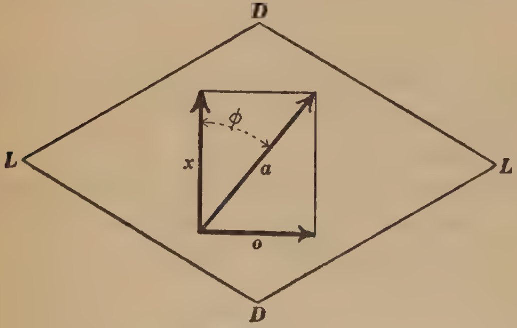

Fig. 194.

$$T = I \cos^2 \phi$$

The intensity *T* of the beam* which is transmitted by the Nicol prism varies, therefore, from a maximum $T = I$ when $\cos^2 \phi$ is unity, to $T = o$ when $\cos^2 \phi$ equals zero.

105. Discussion of the polariscope. Figure 195 shows a polariscope with a Nicol prism polarizer *P* and a Nicol prism analyzer *A* (like Fig. 183), and the figure shows the arrangement for examining the appearance of a crystal plate *CC* in a parallel beam of polarized light. Figure 196 shows a polariscope arranged for examining the appearance of a crystal plate *CC* in a convergent beam of polarized light.

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* Except for a small part which is reflected by the layer of balsam.

The following discussion of the appearance of the crystal plate *CC* in Figs. 195 and 196 is, of course, somewhat incomplete. A very complete discussion of the optical properties is given in Preston's *Theory of Light*, 309–347. See also *A Text Book of Mineralogy*, by E. S. Dana, John Wiley and Sons.

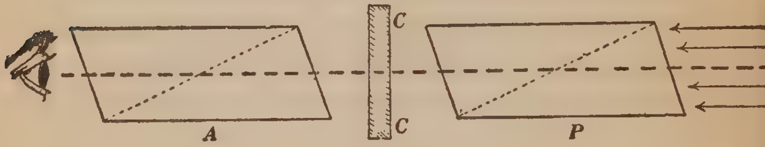

Fig. 195.

106. Discussion of Fig. 195. Looking through the polariscope *without the crystal plate CC*, the observer sees a uniformly bright field if the Nicols are "parallel" or a uniformly dark field if the Nicols are "crossed." With a thin crystal plate *CC* in position (a thin plate of mica, for example) the observer sees a uniformly colored field if the plate is of uniform thickness.

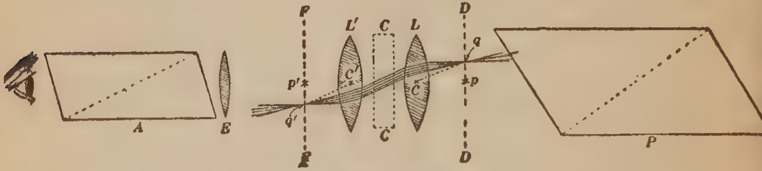

Fig. 196.

Let the vector *i*, Fig. 197, represent the oscillations of the plane polarized light from the polarizer *P* in Fig. 195. On entering the crystal plate the incident beam of polarized light is resolved into two beams* whose oscillation *o* and *e* are at right angles to each other. If *CC* is a plate of Iceland spar then *o* refers to the ordinary beam and *e* refers to the extraordinary beam, and the line *Om* will lie in a principal plane of the crystal plate (a plane which contains the optic axis).

The lines *o* and *e* in Fig. 197 represent the oscillations of the ordinary and extraordinary beams as they enter the crystal plate. Move your pencil point to and fro along *i* and visualize the two components of this to and fro motion along *o* and *e*. The movement along *o* is at *k* when the movement along *e* is at *m*, and we will speak of this as the normal phase relation of the component oscillations along *o* and *e*.

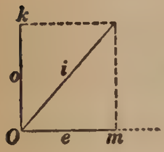

Fig. 197.
Shorter diagonal of polarizer parallel to *i*.

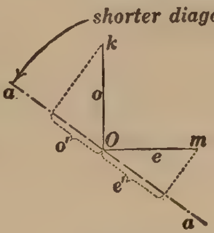

Fig. 198.
Nicols crossed (*aa* at right angles to *i* in Fig. 197).

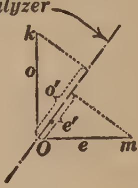

Fig. 199.
Nicols parallel (*aa* parallel to *i* in Fig. 197).

The ordinary and extraordinary beams travel through *CC* at different velocities, and therefore one beam falls a certain distance δ behind the other. This distance δ is called the *relative retardation.*

If the relative retardation δ is an even number of half-wave-lengths, then the oscillation vectors *o* and *e* in Figs.

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* Except when *CC* is a plate of a uni-axial crystal like Iceland spar *with the optic axis at right angles to the polished faces of CC.*

198 and 199, *which refer to the ordinary and extraordinary beams as they emerge from CC*, will have the same normal phase relation as in Fig. 197, that is to say oscillation *o* will be at *k* when oscillation *e* is at *m*, and therefore *o′* and *e′* will be opposite to each other in phase in Fig. 198, and *o′* and *e′* will be alike in phase in Fig. 199.

Now *o′* and *e′* are the components of *o* and *e* which pass through the analyzer and reach the observer's eye. Therefore, assuming *o′* and *e′* to be equal in amplitude (which is not the case in Figs. 198 and 199), it is evident that the two equal and opposite oscillations *o′* and *e′* will annul each other in Fig. 198 whereas the two oscillations *o′* and *e′* in Fig. 199 will intensify each other. That is to say, light of the particular wave-length (which wave-length is contained in 2δ an even number of times) will be transmitted to the observer's eye with great intensity in Fig. 199, but only with very low or even zero intensity in Fig. 198.

If the relative retardation δ is an odd number of half-wave-lengths then the oscillation *o* will be reserved with respect to *e* on emergence from *CC*; that is to say *o* will be at *O* when *e* is at *m* in Figs. 198 and 199. Therefore, in this case *o′* and *e′* will be alike in phase in Fig. 198 and opposite to each other in phase in Fig. 199, and little or no light of the particular wave-length will reach the observer's eye in Fig. 199, whereas light of the particular wave-length will reach the observer's eye in Fig. 198 in great intensity.

Let the student understand that the above discussion can be fully understood by moving a pencil point back and forth along the lines *i*, *o*, *e*, *o′* and *e′*, as stated above, and considering these motions. The whole discussion is purely a matter of kinematics.

The quarter-wave plate. Production of circularly polarized light. Let the oscillation *i* in Fig. 197 be equally inclined to *o* and *e*. Then *o* and *e* are equal in amplitude and at the entrance face of *CC* (to which Fig. 197 refers, as stated above) *o* is at *k* when *e* is at *m*.

Suppose that the crystal plate *CC* in Fig. 195 is of such thickness that the relative retardation δ is one quarter of λ, where λ is the wave-length of the light used. Then on emergence from *CC* the oscillation *o*, Fig. 198, will be at its middle point when *e* is at *m*, and under these conditions the two equal amplitude oscillations *o* and *e* in the beam which emerges from *CC* combine to form a circular oscillation, and the beam which emerges from *CC* will be a circularly polarized beam of light, right-handed or left-handed as the case may be.

107. Discussion of Fig. 196. To understand the action of the crystal plate in Fig. 196 it is important to make note of a simple fact concerning the action of the two lenses *L* and *L'*. The focal point of *L* is at *p* and *DD* is the focal plane of *L*. The focal point of *L'* is at *p'* and *FF* is the focal plane of *L'*. Consider all the rays (of polarized light, of course) which pass through the point *q*; these rays pass through *CC* as a parallel beam, and they are concentrated at the point *q'* by the lens *L'*. The greater the distance *pq* the more inclined the beam of parallel rays through *CC*, and the greater the distance *p'q'*. Therefore if the eye (looking through the magnifying-glass *E*) is focussed on the plane *FF* the brightness and color of each point *p'* of the field of view will depend on the action of the crystal plate *CC* on a parallel beam at a certain definite inclination to the axis of the instrument, as shown in Fig. 196.

The following discussion applies to a plate of Iceland spar with the optic axis at right angles to the two polished faces. Figure 196 shows the crystal plate in the polariscope, and Fig. 200 is a somewhat enlarged view of the crystal plate showing Huygens' ellipsoid and sphere (see Fig. 187).

The parallel beam in *CC* in Fig. 196 is two beams, an ordinary beam and an extraordinary beam with their oscillations *o* and *e* at right angles to each other, as represented by the vector *o* and *e* in Fig. 197, and at the entrance face of the crystal plate the oscillation *o* is at *k* when *e* is at *m*, exactly as in the case discussed in Art. 106.

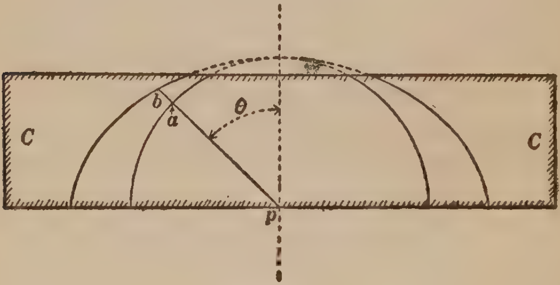

Fig. 200.

The ordinary beam travels slower than the extraordinary beam, and therefore on emergence from *CC* the ordinary beam will have fallen behind the extraordinary beam by the distance δ, exactly as in the case discussed in Art. 106. Now the ordinary and extraordinary beams after emerging from *CC* pass through the lens and then the analyzer *A* transmits only the components o' and e' (see Figs. 198 and 199) to the observer's eye, but the oscillations o' and e' are opposite to each other in phase in Fig. 198 if the relative retardation δ is an even number of half-wave-lengths (in which case o' and e' are alike in phase in Fig. 199), and the oscillations o' and e' are alike in phase in Fig. 198

if δ is an odd number of half-wave-lengths (in which case o' and e' are opposite in phase in Fig. 199).

Now *pa* and *pb*, Fig. 200, represent the velocities of the ordinary and extraordinary rays, respectively, in the direction *pb*, and it is evident from Fig. 200 that the relative retardation δ is greater and greater the more oblique the beam of parallel rays in Fig. 196 (the greater the value of θ in Fig. 200), and for two reasons, namely, (*a*) The greater θ the greater the difference between the two velocities, and (*b*) The greater θ the greater the distance through the crystal plate.

Let us consider only the case where light of one wave-length λ is used in Fig. 196, and let us consider only the case of crossed Nicols, as in Fig. 198, where o' and e' are opposite to each other in phase when δ contains an even number of half-wave-lengths and alike in phase when δ contains an odd number of half-wave-lengths. Then the observer will see a large number of bright and dark circles around the central point p' of the field of view because for increasing distances from p' the relative retardation δ increases steadily, as explained above. If white light is used the observer sees colored circles.

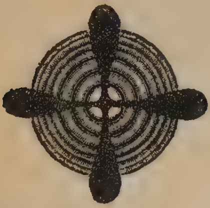

Fig. 201*a*.
Nicols crossed.

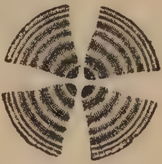

Fig. 201*b*.
Nicols parallel.

The actual appearance of a plate of Iceland spar (cut at right angles to its optic axis) is shown in Figs. 201*a* and 201*b*, and the black cross in Fig. 201*a* and the white cross in Fig. 201*b* are explained as follows:

Figure 202 is a diagram of the field of view of the observer in Fig. 196. The center of the field is at *p′* and the double-headed arrow *A* indicates the direction of oscillation of the polarized light from the polarizer *P* of Fig. 196. Consider any point *q′* of the field of view in Fig. 202. The light which reaches *q′* comes through the crystal plate as ordinary and extraordinary beams whose directions of oscillation are indicated by the short lines *o* and *e* in Fig. 202; *when the point q′ lies on the dotted line CC in Fig.* 202, *the light which reaches q′ comes through the crystal plate wholly as the ordinary beam, the light merges from the crystal plate with its oscillation-character wholly unchanged, and it is not transmitted by the analyzer in Fig.* 201*a, but it is transmitted by the analyzer in Fig.* 201*b*. When the point *q′* lies on the dotted line *DD* in Fig. 202, the light which reaches *q′* comes through the crystal plate wholly as the extraordinary beam, the light

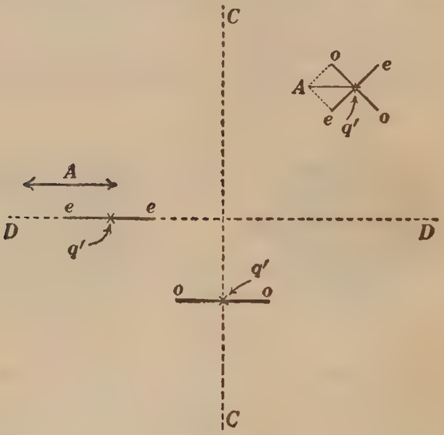

Fig. 202.

emerges from the crystal plate with its oscillation-character wholly unchanged, and it is not transmitted by the analyzer in Fig. 201*a* but it is transmitted by the analyzer in Fig. 201*b*.

Iceland spar is called a *uni-axial* crystal. Sugar crystals, mica crystals and many others are *biaxial*. Figure 203 shows the appearance of a plate of aragonite (a biaxial crystal) in the polariscope of Fig. 196.

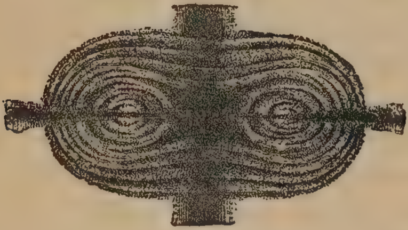

Fig. 203.
Nicols crossed.

108. Rotation of plane of oscillation of polarized light. When a beam of plane polarized light passes through a solution of cane sugar the direction of the wave oscillations is turned about the beam as an axis. Any substance which has this effect on a beam of plane polarized light is said to be "optically active." Thus a solution of cane sugar is optically active, many substances are optically active in the crystalline form, and all substances are optically active in a strong magnetic field. Some substances turn the direction of oscillation clock-wise with respect to an observer towards whom the beam is traveling, and other substances turn the direction in the opposite direction. The former is called *right-handed* rotation and the latter is called *left-handed* rotation. Ordinary cane sugar produces right-handed rotation.

The angle ϕ through which the direction of oscillation of plane polarized light is turned by a solution of cane sugar depends on the wave-length of the light (being greatest for violet and least for red) and for a given wave-length it is given by the equation

$$\phi = ksl \qquad \text{(i)}$$

where s is the sugar content of the solution, l is the distance traveled in the solution, and k is a constant. When sodium light is used, when s is expressed in grams of sugar per cubic centimeter of solution, when l is expressed in centimeters, and when ϕ is expressed in degrees, then the value of k is 6·65.

The *saccharimeter** is a polariscope between the polarizer and analyzer of which is placed a long tube with flat glass ends. A solution of sugar is placed in this tube, and the analyzer is arranged to permit the measurement of the angle ϕ, whence, knowing k and l, the strength of the solution may be calculated from equation (i).

109. The polarization of sky light. The light from the clear sky is sunlight diffused by extremely small particles of dust and perhaps also by the molecules of the air, and sky light is more or less polarized. The explanation of this effect is as follows: Fig. 204 shows an extremely small particle p in a beam of sunlight. The wave oscillations in the beam of sunlight are at right angles to the beam (at right angles to B); if these oscillations are imparted to p at all it is evident that p will oscillate in a plane at right angles to B; these oscillations of p as seen by an observer looking sidewise at p will evidently lie in the straight line *bb*; and if *p* sends any light at all to the observer the wave oscillations in this light will be parallel to *bb* or lie in a plane at right angles to the plane of the paper in Fig. 204 and containing the line *bb*. That is to say, the light sent out sidewise from *p* will be plane polarized.

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* The saccharimeter of Laurent is described on pages 68–71 of Vol. III of Franklin, Crawford and MacNutt's *Practical Physics*. A very good discussion of various forms of saccharimeter is given by Otto Lummer on pages 1172–1188 of Müller-Pouillet's *Lehrbuch der Physik*, Vol. II, Part I, Braunschweig, 1897.

This effect can be studied by filling a beaker with water containing extremely small particles in suspension, illuminating the milky water from above, looking at the illuminated water sidewise through a Nicol prism (or any form of analyzer), and turning the prism. When the shorter diagonal of the end-face of the prism is vertical (parallel to the illuminating beam) the brightness of the illuminated milky water is a minimum, and when the shorter diagonal of the end-face of the prism is horizontal the brightness of the illuminated milky water is a maximum. The milky water is best made by stirring a few drops of very dilute alcoholic solution of gum mastic or rosin in clear water. A milky syrup is made by stirring a few drops of the alcoholic solution in clear syrup.

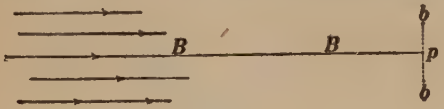

Fig. 204.

If a beam of plane polarized light is passed downwards into a beaker of milky water the light which is diffused by the fine suspended particles will be given off mostly in directions at right angles to the direction of oscillation of the plane polarized beam.

A very beautiful experiment is as follows: A large glass tube with flat glass ends is filled with milky syrup, and the milky syrup is illuminated by a beam of plane polarized light which enters one end of the tube.

Looking at the tube from one side, the milky syrup sends little or no light to the eye where the direction of the wave oscillations of the illuminating beam *lies in a plane including the eye*, and the milky syrup sends a maximum of light to the eye where the direction of the wave oscillations is at right angles to this. This will be understood by a careful reading of the discussion of Fig. 204.

But the direction of wave-oscillation is rotated more and more as the polarized light travels along the tube, and the amount of rotation is greatest for violet and least for red. Therefore the luminosity of the milky syrup as seen from one side consists of a whole series of helices (or screw-lines) of color, and the pitch of these screw-lines will be least for violet and greatest for red.

LEADING QUESTIONS

L. Q. 65. How do the particles of a rope move when a train of plane polarized waves travels along the rope? When a train of circularly polarized waves travels along the rope? When a train of elliptically polarized waves travels along the rope?

L. Q. 66. Holding one end of a stretched rope in the hand, how would you move the hand so as to produce waves on the rope of the kind that exist in ordinary unpolarized light? How would you move the hand so as to produce plane polarized waves on the rope? How would you move the hand so as to produce circularly polarized waves on the rope?

L. Q. 67. A plate of tourmaline is dichroic. What does this mean?

L. Q. 68. The index of refraction of water is 1·33. At what angle (between the beam and the normal to the water surface) must a beam of light strike the water surface in order that the reflected beam may be completely polarized (plane polarized)? What is the direction of the wave oscillation in the reflected beam?

L. Q. 69. Why is the ray *o* or the ray *O* in Fig. 186 called the ordinary ray? Why is the ray *x* or *X* called the extraordinary ray?

L. Q. 70. What is meant by an axis of symmetry in a crystal?

L. Q. 71. What is the optic axis in a crystal of Iceland spar? This question is to be answered by considering the geometry of the crystal.

L. Q. 72. What is a principal plane in a crystal of Iceland spar?

L. Q. 73. Imagine a center of disturbance (optical) in a crystal of Iceland spar. What kind of waves would pass out from such a point?

L. Q. 74. Make sketches of Figs. 187, 189, and 191, and make a number of double-headed arrows showing the directions of oscillation of the ellipsoidal wave or wavelet in each.

L. Q. 75. Make a sketch of Fig. 190, and make a number of double-headed arrows showing the directions of oscillation of the spherical wave or wavelet.

L. Q. 76. Specify the direction of a principal plane in each of the figures 187 to 191.

L. Q. 77. Why is the ray *o* completely reflected by the layer of Canada balsam in Fig. 193?

L. Q. 78. The sun is on the horizon in the west. What is the direction of the wave oscillations in the polarized light from the sky overhead?

PROBLEMS

71. Taking the intensity of the light which is transmitted through *A* and *P* in Fig. 183 as unity when the two Nicols are "parallel," find the intensity of the transmitted light when the analyzer *A* is turned 30° from the "parallel" position.

72. A plate of Iceland spar 0·02 millimeter thick with the optic axis parallel to the polished surfaces of the plate is placed in the polariscope of Fig. 195 with the analyzer in the "crossed" position. What wave-lengths within the limits of the visible spectrum are cut off by the analyzer? The ordinary and extraordinary indices of refraction of Iceland spar (referred to air) are given in Art. 102. Ans. 68·8, 57·3, 49·1 and 43·0 millionths of a centimeter.

Note. Of course, the indices of refraction of Iceland spar vary somewhat with the wave-length of the light, but this variation is to be neglected in this and the following problems.

73. Find the thickness of a quarter-wave plate for sodium light ($\lambda = 5890 \times 10^{-8}$ cm.), the plate being cut from a crystal of Iceland spar with the optic axis of the crystal parallel to polished faces of the plate.

74. A solution of cane sugar in a tube 50 centimeters long rotates the plane of polarization of sodium light through 8·31 degrees of angle. What is the amount of sugar in each cubic centimeter of the solution?

CHAPTER XI

COLOR*

110. Sensations of brightness and sensations of color. A beam of light which falls upon the retina produces two distinct sensations, namely, a *sensation of brightness* and a *sensation of color.* The intensity of the sensation of brightness depends upon the physical intensity of the light,† and the character and vividness of the sensation of color depends upon the wave-length of the light, or when the light is not homogeneous, upon the relative intensities of the various wave-lengths which are present in the light.

111. The flicker photometer. The flicker photometer is a device for eliminating to some extent the error of setting of a Bunsen photometer which is due to differences in color of the lamps which are being compared, and the following is the principle upon which the elimination of color error is based. When one looks at a thing such as a photometer screen two sensations are produced, namely, a sensation of brightness and a sensation of color, as stated in Art. 110. Both of these sensations persist for an appreciable interval of time after stimulation ceases, but the sensation of color persists much longer than the sensation of brightness. Therefore if the two sides of the photometer screen are brought into the same field of view in rapid succession (with high frequency of interchange), the *color sensation* produced by the two sides of the screen and also the *brightness sensation* produced by the two sides of the screen will both become perfectly steady (devoid of flicker), whereas a much lower frequency of interchange will suffice to give a steady color sensation but leave a flickering sensation of brightness *unless the two sides of the screen have the same brightness.* Therefore if we use a frequency of interchange just suffi-

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* The most complete treatise on color from both the physical and physiological points of view is to be found in Helmholtz's *Handbuch der Physiologischen Optik,* pages 275–384.

An extremely interesting discussion of brightness and color effects in their bearing upon painting is to be found in Helmholtz's popular lecture on The Relation of Optics to Painting, see footnote to Art. 31 of Chapter III.

An interesting book is Ogden Rood's *Text Book of Color or Modern Chromatics,* New York, 1881.

† Also to some extent on the wave-length, see Art. 112.

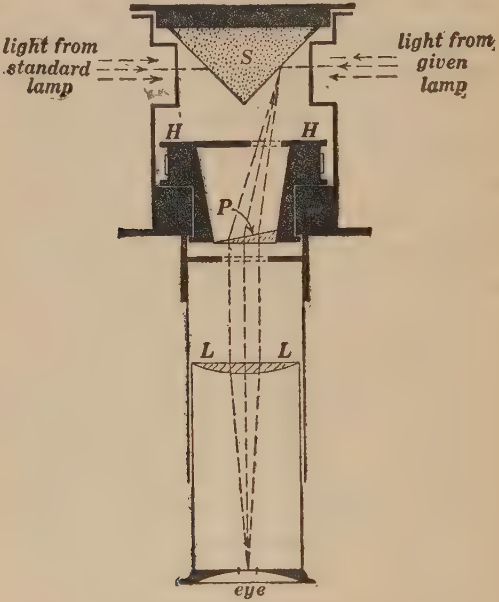

Fig. 205.

cient to give a steady color sensation, the two sides of the photometer screen can be brought to equality of brightness by adjusting the photometer until the brightness-flicker disappears. The various flicker photometers differ in the arrangement used for bringing about the rapid interchange of the two sides of the photometer screen in the same field of view, and the most satisfactory device is perhaps that due to Marten.* The two faces of a prism of white plaster *S*, Fig. 205, are illuminated by the standard lamp and a given lamp respectively. The eye is focussed upon one of the illuminated faces of the plaster prism by means of the lens *LL*, and the thin glass prism *P* which is carried in a rotating holder *HH* directs one's vision to one face and then to the other face of the plaster prism in rapid succession.

Another form of flicker device which is quite satisfactory is that of Simmance and Abady.†

112. Luminosities of various parts of the spectrum. The different parts of the spectrum differ greatly in brightness, and by no means in proportion to the energy of the different parts. The ordinates of the dotted curve in Fig. 206 represent the energy or heating intensities of the various parts of the spectrum of the light from a kerosene lamp, and the ordinates of the full-line curve represent the degrees of brightness of the various parts of the spectrum to the normal eye. The abscissas represent wave-lengths.‡

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* This flicker photometer is manufactured by Schmidt and Haensch, of Berlin.

† The Simmance-Abady flicker sight-box is described on pages 491–492 of Vol. I, *Johns Hopkins University Lectures on Illuminating Engineering*, Johns Hopkins Press, 1911.

‡ A simple method for determining this luminosity curve, due to Tufts, is described in Franklin, Crawford and MacNutt's *Practical Physics*, Vol. III, pages 26–27.

113. Color sensations due to homogeneous light. Light of one wave-length is called homogeneous light as explained in Art. 81, and the various wave-lengths of homogeneous light produce the familiar series of colors of the spectrum as seen in a spectroscope. Newton recognized and named seven colors in the spectrum: red, orange, yellow, green, blue, indigo, and violet. As a matter of fact, however, about 150 steps are made in going through the spectrum from one tint to the next which can barely be distinguished from it. That is to say, there are about 150 distinguishable tints in the spectrum. The most vivid colors in the spectrum are the extreme red, the green, and the blue or violet.

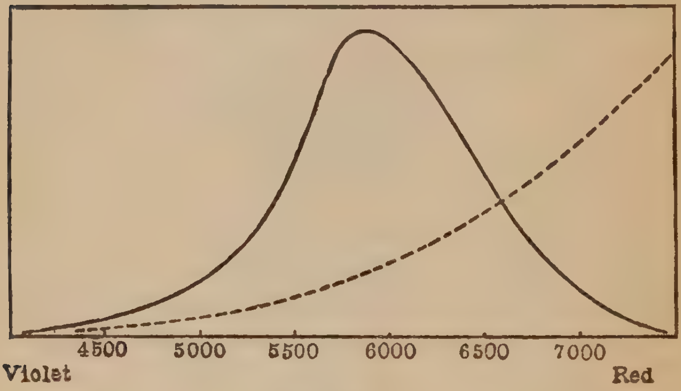

Fig. 206.

114. Color sensations due to mixed light. It is evident that there is a possibility of an infinite variety of mixtures of the various wave-lengths of light, according to the relative intensities of the various wave-lengths that are present in the mixture, and the number of distinguishable color sensations which may be produced by various light mixtures is, according to Titchener,* about 30,000, and, according to Rood,† a much larger number.

White light. Sun light, or any light approaching sun light in composition, is called *white light*. The sensation produced by such light, aside from complications growing out of contrast effects, is called *white*.

Saturated and diluted colors. Mixed light which contains a great excess of one wave-length or narrow group of wave-lengths generally produces a vivid sensation of color. Such a color is sometimes called a *saturated color*. Mixed light which approaches white light in composition, having only a slight excess of one wave-length or group of wave-lengths, gives a pale sensation of color. Such a color is sometimes called a *diluted color*.

115. The cause of colour in natural objects. Colored objects occurring in nature owe their color to the fact that they send to the eye light of which the composition (relative intensities of the different wave-lengths) differs more or less widely from the composition of white light.

Examples. (*a*) Hot gases and vapors give off light which differs widely in composition from white light. Thus, most of the color effects in fireworks are produced by the use of salts of the various metals. These salts are vaporized and the hot vapors give off brilliantly colored lights.

(*b*) Many substances reflect certain wave-lengths in great excess. Thus the ordinates of the curve in Fig. 207 represent the fractional parts of the various homogeneous components (wave-lengths) of ordinary white light which are reflected by powdered ultramarine-blue. About 24 per cent of the light in the neighborhood of Fraunhofer's *G* line is reflected, whereas about 3 per cent of the light in the neighborhood of the *E* line is reflected. The measurements from which Figs. 207 and 208 were plotted were made by means of the spectrophotometer. See Art. 89 of Chapter VIII.

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* Titchener, *An Outline of Psychology*, page 66.

† Rood, *A Text Book of Color*, Chapter IX.

(*c*) Many substances such as colored glass transmit certain wave-lengths in great excess. Thus the ordinates of the curve in Fig. 208 represent the fractional parts of the various homogeneous components (wave-lengths) of ordinary white light which are transmitted by a given solution of potassium chromate.

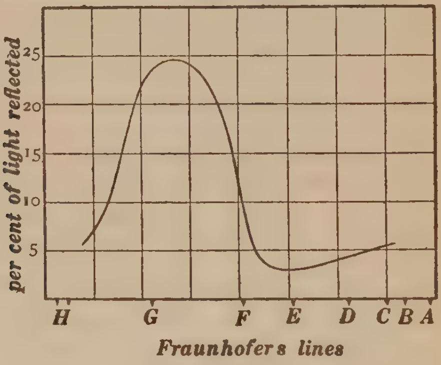

Fig. 207.
Light reflected by ultramarine-blue.

116. Mixing of colors. It has been known since the earliest days of painting that two colors blend completely and give a single resultant color or tint when mixed.

To understand the difference between the *mixing of colored lights*, and the *mixing of pigments* let us consider the following example.

Let us think of white light as containing 100 parts of red light, 100 parts of green light, and 100 parts of violet light.

Given a pane of yellow glass which transmits one half of the red (50 parts), one quarter of the green (25 parts), and not any violet.

Given a pane of blue glass which transmits not any of the red, one quarter of the green (25 parts), and one half of the violet (50 parts).

(*a*) Imagine the two panes to be set side by side in a window frame, then the light which comes into the room through the yellow pane would be mixed with or added to the light which comes into the room through the blue pane, and we would have in the room 50 parts red light, 50 parts green light, and 50 parts violet light, which would be *white light*, somewhat weakened, of course.

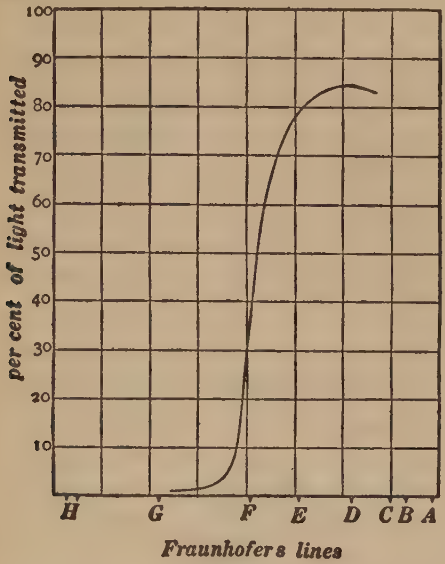

Fig. 208.

Light transmitted by a solution of potassium chromate.

(*b*) Imagine the two panes to be set one over the other as one double-thickness pane. Then the two panes together would transmit no red light and no blue light, but they would transmit $\frac{1}{4}$ of $\frac{1}{4}$ or $\frac{1}{16}$ of the green which would be $6\frac{1}{4}$ parts of green light. That is to say, we would have green light in the room.

(*a*) Yellow and blue lights when mixed (added) give white.

(*b*) Yellow and blue glasses laid over each other give green.

Something similar to (*b*) takes place when pigments are mixed. Thus a yellow pigment mixed with a blue pigment gives a green pigment.

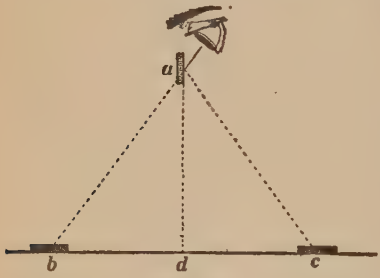

Fig. 209.

The simplest device for mixing two colored lights is that which is shown in Fig. 209, in which *a* is a plate of unsilvered glass, and *b* and *c* are bits of colored paper or colored paint. The light from *b* passes through the glass plate *a* into the eye, and the light from *c* is reflected by the glass plate into the eye, as indicated by the dotted lines.

The most convenient arrangement, perhaps, for mixing colored lights is the *color top*. This consists of a rotating spindle upon which are mounted disks of colored paper which are slitted in such a way that any desired sector of the face of each disk may be exposed to view. When this composite disk is rotated rapidly the colors blend and give a single sensation of color, the tint of which may be modified at will by varying the amounts of the respective disks that are exposed.

Complementary colors. Two colors (colored lights) which produce white light when mixed (added) are called *complementary colors.* Thus yellow and indigo-blue are complementary colors.

117. Matching of colors by mixtures. Two-color vision and three-color vision. *Any color may be matched by a mixture, in proper proportions, of a saturated red light, a saturated green light, and a saturated violet light.* This is an experimental fact which was discovered by Thomas Young.

For some persons (red blind) any color may be matched by a mixture, in proper proportions, of a saturated green light and a saturated violet light. For other persons (green blind) any color may be matched by a mixture, in proper proportions, of a saturated red light and a saturated violet light. Persons for whom any color may be matched by mixing two saturated colors are said to have *dichroic* or *two-color vision.* Persons for whom the matching of any color requires the mixing of three saturated colors are said to have *trichoic* or *three-color vision.* Dichroic vision is commonly called *color blindness.* About 4 per cent of the male population and about four-tenths of 1 per cent of the female population of the civilized world are color blind.

118. The Young-Helmholtz theory of color. The fact that any color can be matched by a proper mixture of three saturated colors led Thomas Young, in 1801, to infer the existence of three primary color sensations. Helmholtz attributed each of these primary sensations to a distinct set of nerves in the retina of the eye. The nerves which upon excitation give the primary sensation of red are called the *red nerves*, those which upon excitation give the primary sensation of green are called the *green nerves*, and those which upon excitation give the primary sensation of violet are called the *violet nerves*. Simultaneous excitation of all three sets of nerves gives a blended sensation the character of which depends upon the relative intensities of excitation of the respective sets of nerves.

A person having dichroic vision has, according to the Young-Helmholtz theory, only two sets of color nerves, and therefore only two primary color sensations. Persons who do not have the primary sensation of red are said to be *red blind* ; and persons who do not have the primary sensation of green are said to be *green blind*. No clearly defined case of *violet blindness* has ever been found.

The Young-Helmholtz theory of color is not generally accepted. Physiologists are inclined to reject it for lack of microscopical evidence of the existence of the three sets of nerves, and psychologists are inclined to reject it mainly because of the difficulty of explaining the great number of distinguishable color sensations by the varying intensities of sensation of three sets of nerves. The theory gives, however, a very clear representation of the experimental facts of color mixing and very satisfactory explanations of contrast effects and color blindness.

119. Sensitiveness of the color nerves to lights of different wave-length. When a given sensory nerve is excited by different means the sensation is always the same, as explained in Art. 2. Thus the excitation of the red, green or violet nerves always gives sensations of red, green or violet, respectively, however the excitation may be produced. The ordinates of the curves in Fig. 210 represent the relative degrees of sensitiveness of the respective sets of color nerves to different wave-lengths of light. These curves are from measurements made by Koenig

As these curves show, every wave-length in the spectrum can excite the red nerves and in some degree give the sensation of red. The green nerves can be excited more or less by any wave-length between Fraunhofer's *C* line and *G* line; and the violet nerves can be excited more or less by any wave-length between the *E* line and the *H* line.

Any particular wave-length of light excites all three sets of nerves more or less, and produces a blended sensation.

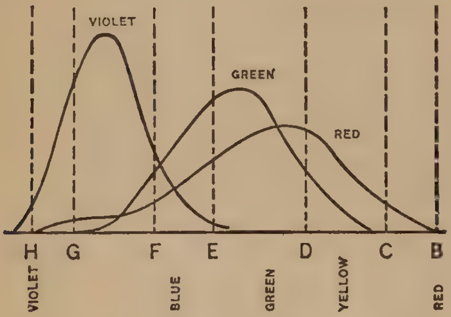

Fig. 210.

120. Contrast effects. Two complementary colors placed side by side tend to intensify each other as the eye glances from the one to the other. Any two colors placed side by side, if they are at all different, tend to become complementary in appearance as the eye glances from one to the other. This *contrast effect,* as it is called, may be shown very strikingly by placing two small pieces of pale green paper exactly alike, one on a large sheet of red paper and the other on a large sheet of blue paper. The two pieces of green paper are so greatly changed by the opposite contrasts that one can scarcely believe that they are physically alike. A bit of white or grayish paper placed upon a broad sheet of brilliantly colored paper assumes a very distinct hue complementary to the surrounding color.

The explanation of contrast effects in terms of the Young-Helmholtz theory of color is as follows: White light affects all three sets of color nerves in certain relative proportions. A colored light also affects all three sets of nerves, but the vividness of the color sensation which is produced depends upon the preponderating excitation of one or two sets of nerves. When one looks at a brilliant color, one or two sets of colored nerves become more or less fatigued, and, under these conditions, a neutral tint produces a decreased excitation of these fatigued nerves or a relatively greater excitation of the unfatigued set of nerves, thus tending to produce the complementary color sensation.

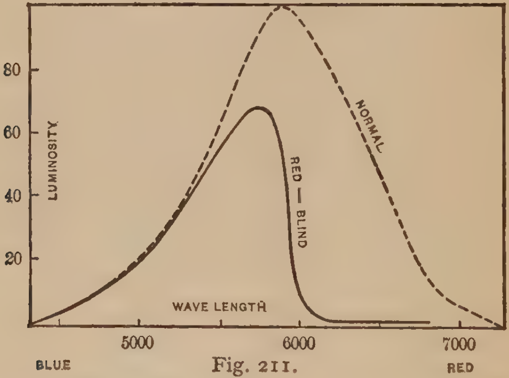

Fig. 211.

121. **Color blindness.** Color-blind persons show marked peculiarities in their sensations of brightness and color. Thus the ordinates of the dotted curve in Fig. 211 represent the relative degrees of brightness of the various parts of the spectrum of gas light as seen by

the normal eye (the dotted curve in Fig. 211 is the same thing as the full-line curve in Fig. 206), and the ordinates of the full-line curve in Fig. 211 represent the relative degrees of brightness of the various parts of the spectrum of gas light to a red-blind person.*

Color sensations due to homogeneous light. Figure 212 shows roughly the appearance of the spectrum to persons with normal trichroic vision, to green-blind persons and to red-blind persons respectively. The red and violet nerves of a green-blind person are both excited more or less by any wave-length of light, and that particular wave-length which excites these two sets of nerves in the same proportion that ordinary white light excites them gives to the green-blind person the sensation which he calls white. This wave-length lies between the *E* and *F* lines of Fraunhofer according to Fig. 210 and as indicated in Fig. 212.

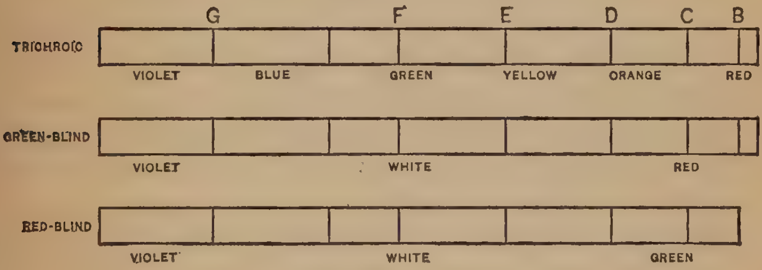

Fig. 212.

Similarly the red-blind person sees what he calls a "white" region in the middle part of the spectrum.

Color sensations due to mixed light. Color-blind persons show marked peculiarities of color sensations due to mixed lights. These peculiarities are very complicated, in fact, they are hardly the same for any two persons. The following article which describes the Holmgren test for color-blindness, gives some idea of the peculiar color sensations of color-blind persons.

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* From measurements made by Ferry, *American Journal of Science*, Vol. 44, 1892.

122. The Holmgren test for color blindness. All systems of signalling upon railways and at sea depend more or less upon the recognition of colored lights, and consequently the character of the color sense of employés is a matter of vital importance. It is fortunately possible to detect dichroic vision with certainty even when the existence of the peculiarity is unsuspected by the subject, or when the subject attempts to conceal the matter. The simplest method of performing such tests was invented by Holmgren, of Upsala, after the occurrence (in Sweden) of a dreadful railway accident due to the color blindness of an employé. The Holmgren apparatus consists simply of a collection of colored worsteds which includes a large variety of colors of every degree of saturation and a number of neutral grays of various degrees of brightness. There are also three larger skeins which are called the *confusion samples*. One of these is a pale green, corresponding very nearly in tint (but, of course, not in saturation) to the portion of the spectrum which appears white to both red-blind and green-blind persons, as described in Art. 121. Another confusion sample is pale magenta, which is a blend of red and violet with much white, and the third confusion sample is a brilliant red.

Both red-blind and green-blind persons invariably select a variety of neutral or gray worsteds as corresponding most closely with the pale green confusion sample.

Red-blind persons invariably select violets and blues as corresponding most closely with the magenta confusion sample.

Both red-blind and green-blind persons are inclined to select brilliant greens as well as brilliant reds in matching the bright red confusion sample.

LEADING QUESTIONS

L. Q. 79. The brightest part of the spectrum of the light from a kerosene lamp is in the orange-yellow. This does not mean that the energy intensity of the spectrum is a maximum in the orange-yellow. Explain.

L. Q. 80. What is meant by white light? The light from a lamp can be made to approximate to the quality of daylight by cutting off a large portion of the red and smaller portions of the yellow and green from the light emitted by the lamp. How can this be done?

L. Q. 81. What is meant by a saturated color? What is meant by a diluted color? Give examples.

L. Q. 82. Colored lights are produced in nature by selective emission, by selective reflection, and by selective absorption. Explain the meaning of each of these terms and give examples illustrating each effect.

L. Q. 83. What is meant by three-color vision? What is meant by two-color vision? Explain each in terms of the Young-Helmholtz theory.

L. Q. 84. Explain, in the terms of Young-Helmholtz theory, why a piece of gray paper lying on a sheet of red paper appears to be greenish in color.

L. Q. 85. On the basis of Fig. 210 explain why the portion of the spectrum between *F* and *E* in Fig. 212 appears "white" to a red-blind or green-blind person.

CHAPTER XII

TONES AND NOISES. LOUDNESS, PITCH AND QUALITY

123. The methods of optics and the method of acoustics. In the study of light we have been concerned chiefly with the phenomena which are associated with transmission, such as the phenomena of reflection, refraction and diffraction. In the study of sound, on the other hand, we are concerned chiefly with the mode of generation of sound waves and with the effects which are produced by sound waves when they strike a body. The phenomena of reflection, refraction and diffraction are as real in the case of sound waves as in the case of light waves, but these phenomena of sound are not of great importance.

The more advanced study* of optics is concerned with the mode of generation of light waves in luminous bodies, and with the effects of light waves upon bodies or substances upon which light waves fall, but these things are beyond the scope of this text.

124. Noises and musical tones. Sound, in the physical sense of the word, consists of waves in the air which travel from the sound producing body to the ear, as stated in Arts. 4 and 5; these waves are produced by quick movements of the sounding body, and when they fall on the ear they produce the sensation of sound.

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* The branches of optics here referred to are spectrum analysis including much recent work on the absorption and emission of light and of X-rays. Also the theory of refraction, including the theory of dispersion, has to do with the character of the disturbance produced in a body which receives a beam of light. The theory of dispersion is discussed in Drude's *Theory of Optics*, pages 382–389.

When a sounding body performs a succession of similar movements to and fro, its motion is said to be periodic, and it produces a succession of similar sound waves, or a *wave-train*. The sensation corresponding to such a wave-train is called a *tone*.

Sound sensations which cannot be classified as tones or combinations of tones are called *noises*. For example, rattling noises are due to irregular successions of sharp clicks, each of which sends a single wave pulse to the ear. Hissing and roaring noises are due to complex and rapidly varying combinations of tones. In the case of hissing noises the tones, which may be few in number, are of a very high pitch,* and in the case of roaring noises the tones are usually numerous and of lower pitch.

Musical tones are generally accompanied by characteristic noises. Thus, the whispering noises of the breath and the sounds of the consonants used in articulation accompany the musical tones of a singer; and the faint noises produced by the fingers, keys and pedals always accompany piano music. Many noises, on the other hand, are accompanied by distinctly audible musical tones; thus, a light hammer-blow upon a floor or upon a piece of furniture produces a musical tone of short duration which is often easily distinguishable.

125. Loudness of tones. It is a familiar fact that the sound emitted by a vibrating body, such as a guitar string, increases in loudness with increase of the amplitude of the vibrations (by amplitude is meant half the distance through which the middle of the string swings to and fro).

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* See Art. 126.

When sound waves travel through the air, any given particle of the air oscillates to and fro through a certain amplitude, and the amplitude of the air oscillations increases with the amplitude of the given vibrating body which produces the sound waves.* *The loudness of a tone depends upon the amplitude of the air oscillations.*

126. Pitch of tones. Consider a musical instrument such as the piano or harp. The short strings have a greater frequency of vibration than the longer strings, and that quality of a musical tone which depends upon the frequency of the vibrations is called *pitch*. The pitch of a tone is high or low according as the vibration frequency is great or small. Two vibrating bodies which give tones of the same pitch are said to vibrate in unison.

Determination of pitch.† The direct determination of pitch is accomplished by counting the number of vibrations in a given time. The siren is sometimes used for this purpose. It consists of a circular metal disk mounted on a shaft which is geared to a revolution counter. The disk has one or more circular rows of equidistant holes, it rotates near to the wall of a chamber containing air under pressure, and the holes in the disk come before apertures in the wall in rapid succession. The puffs of air thus produced blend into a tone, the pitch (number of vibrations per second) of which is known from the observed speed of the disk and the known number of holes in the row. The disk is sometimes driven by an electric motor and sometimes by the action of the puffs of air. In the latter case the holes in the disk are inclined like the vanes of a windmill.

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* A vibrating string may oscillate through a wide amplitude and emit a much weaker sound than is emitted by a board or plate which vibrates through a much smaller amplitude, so that the amplitude of the vibrating body *alone* does not determine the amplitude of the air oscillations.

† See Poynting and Thomson's *Sound*, pages 36–47.

Standards of pitch. A vibrating body which has an invariable frequency of vibration may have its pitch accurately determined once for all, and the pitch of any sound may then be determined by comparison with this standard. Standards of pitch are usually in the form of tuning forks (see Art. 141).

Pitch limits of audibility. When a vibrating body has a frequency less than about 30 complete vibrations per second it does not produce a sensation of tone, but the ear perceives the separate impulses as a fluttering noise. When a vibrating body has a frequency greater than about forty thousand complete vibrations per second it does not produce any sensation of sound at all.

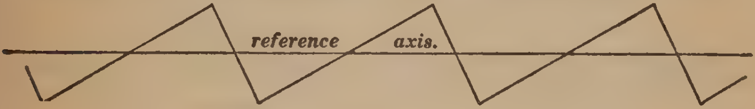

Fig. 213.

127. Tone quality or timbre. It is a familiar fact that one can easily distinguish musical tones produced by different musical instruments independently of differences in loudness and pitch. Thus, a note of the same pitch and loudness produced by a singer and by a violin are so entirely different that there is no difficulty whatever in distinguishing the one from the other. This difference in quality depends upon the character of the oscillations of the vibrating body (or upon the character of the oscillations of the air particles in the sound waves which are produced by the vibrating body). Thus, a point on a violin string usually moves slowly in one direction and more rapidly in the opposite direction, like the up and down motion which would have to be given to a pencil point to trace the zigzag line in Fig. 213 with the paper moving uniformly to the left.

Figure 214 shows the character of the vibrations which correspond to the various vowels when sung by a baritone voice. It must not be thought from this figure, however, that a given vowel sound is produced by a characteristic type of oscillation. This matter is quite fully discussed in Art. 147.

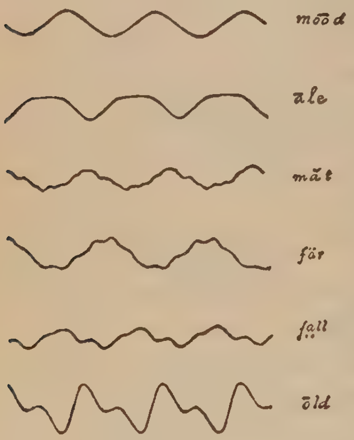

Fig. 214.

128. Simple vibrations and compound vibrations. When a particle moves to and fro along a straight line performing harmonic motion* its vibrations are called *simple vibrations*. When the periodic to and fro motion of a particle is *not* harmonic its vibrations are called *compound vibrations*. The vibrations of a pendulum bob (through a small amplitude) and the vibrations of the tip ends of a tuning fork are examples of simple vibrations. The vibrations of a reed or spring which is slowly raised and quickly dropped (by the cogs of a rotating wheel, for example) are compound vibrations.

Graphical representation of simple and compound vibrations. Imagine a pencil point p in Fig. 215 to vibrate up and down along the line AB, and imagine the paper to move at uniform velocity to the right. Then the point *p* will trace a curve. If the vibrations of *p* are *simple* the curve *cc* will be a curve of sines. If the vibrations of *p* are *compound* the curve *cc* will be a periodic curve, that is, each section of the curve will be exactly like every other section, but it will not be a curve of sines.

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* See *Lessons in Mechanics*, Arts. 44, 45, and 46.

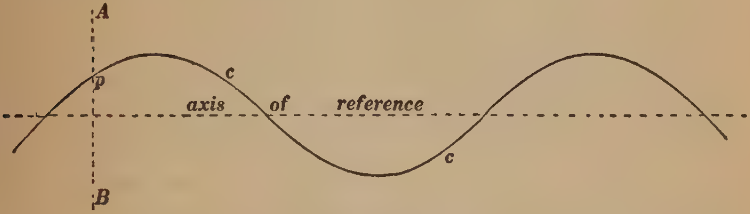

Fig. 215.

The curve in Fig. 215 is a curve of sines and it represents simple vibrations. The curve in Fig. 216 is a periodic curve (not a curve of sines), and it represents compound vibrations. The curve in Fig. 213 and all but the first curve in Fig. 214 represent compound vibrations.

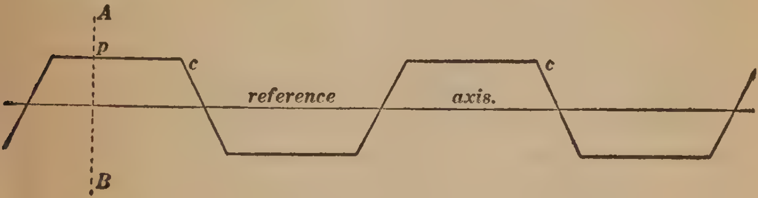

Fig. 216.

Definitions. The number of complete vibrations (round-trip vibrations) per second is called the *frequency* of the vibrations. The duration of one complete vibration is called the *period* of the vibrations. One-half of the distance through which the vibrating particle moves to and fro is called the *amplitude* of the vibrations.

129. Superposition of simple vibrations. Fourier's theorem. A particle may perform simultaneously two or more distinct vibratory movements. In such a case the vibrations are said to be superposed. Thus, if the hand be moved slowly up and down and if, at the same time, one finger be moved quickly up and down, the moving finger would trace a curve similar to the third or fourth or fifth curve in Fig. 214. In this example one vibration is assumed to be of low frequency and the other of high frequency in order that the movements may not be too greatly confused; as a matter of fact, however, any number of vibratory movements, whatever their amplitudes and frequencies, may be performed by a particle simultaneously.

Fourier's theorem. Any periodic vibration of frequency n, however complicated, may be matched by superposing simple vibrations of which the frequencies are n, $2n$, $3n$, $4n$, and so on, provided the respective amplitudes are properly chosen. That is to say, any periodic vibration of frequency n may be thought of as composed of a series of simple vibrations of which the frequencies are n, $2n$, $3n$, $4n$, and so on. *It is for this reason that non-harmonic vibrations are called compound vibrations.*

Application of Fourier's theorem to the explanation of tone quality or timbre. Aside from accompanying noises which frequently characterize musical tones, the difference in quality between two musical tones depends upon the character of the oscillations. If the oscillations are simply harmonic, the tone is called a *simple tone.* If the oscillations are compound, the tone is called a *compound tone* or *clang.* According to Fourier's theorem, a compound tone is the blending together of a series of simple tones of which the frequencies are n, $2n$, $3n$, $4n$, and so on. The tone of lowest frequency n is called the *fundamental* and the other tones are called *overtones* The overtones in the note which is produced by a piano string can be heard easily and distinctly by a musically trained ear. See Art. 146.

130. Simple wave-trains and compound wave-trains. A periodic disturbance sends out a succession of similar waves through the surrounding air. A succession of similar waves is called a *wave-train*, and the distance between similar parts of two successive waves in a train is called the *wave-length* λ of the train. See Art. 11 of Chapter II.

When a wave-train passes through the air each particle of the air oscillates to and fro as the wave-train passes by. When this oscillatory motion is harmonic the wave-train is called a *simple wave-train*, when this oscillatory motion is non-harmonic the wave-train is called a *compound wave-train*.

A simple wave-train is represented graphically by a curve of sines, a compound wave-train is represented graphically by a periodic curve which is *not* a curve of sines. Thus the sine curve in Fig. 215 may be thought of as representing a simple wave-train, and the periodic curve in Fig. 213 or the periodic curve in Fig. 216 may be thought of as representing a compound wave-train.

Fourier's theorem as applied to wave-trains. Any periodic curve, such as shown in Fig. 213 or 216 for example, can be exactly matched by superposing a series of sine curves of which the wave-lengths are λ, $\lambda/2$, $\lambda/3$, $\lambda/4$ and so on, where λ is the wave-length of the given periodic curve. Therefore a wave-train which is represented by any periodic curve whatever may be thought of as the combination or superposition of a series of simple wave-trains whose wave-lengths are λ, $\lambda/2$, $\lambda/3$, $\lambda/4$, etc It is for this reason that a wave-train which is represented by a periodic curve which is *not* a curve of sines is called a compound wave-train.

A body, like a tuning fork, which performs simple harmonic vibrations sends out a *simple train* of sound waves. A body, like the reed of a clarinet, which performs compound vibrations sends out a *compound* train of sound waves.

LEADING QUESTIONS

L. Q. 86. What is the physical difference between a noise and a musical tone?

L. Q. 87. Two musical tones differ in loudness. What is the physical basis of this difference?

L. Q. 88. Two musical tones differ in pitch. What is the physical basis of this difference?

L. Q. 89. What is the physical difference between a sustained violin tone and a sustained clarinet tone of the same loudness and pitch?

L. Q. 90. Why are the vibrations which are represented by the curve in Fig. 213 called compound vibrations?

L. Q. 91. State Fourier's theorem as applied to compound vibrations?

L. Q. 92. State Fourier's theorem as applied to a compound wave-train. Give two simple wave-trains whose wave-lengths are 6 inches and 3 inches, respectively; whose amplitudes are 2 inches and 1 inch, respectively; and which are so related in phase that they both cross the reference axis from — to + at the same point. Plot the two wave-trains separately, and plot their resultant.

PROBLEMS

75. The disk of a siren is driven at a speed which brings its tone into unison with the tone of a tuning fork. The speed of the disk is then observed to be 1820 revolutions per minute, and the row of holes in the siren disk contains 20 equidistant holes. What is the frequency of vibration of the tuning fork?

Note. The sound of a siren is very rich in overtones and it is sometimes very difficult to tell whether a given tone is in unison with the fundamental tone of the siren or one of its overtones.

76. Two tones of which the frequencies are 275 and 300 vibrations per second respectively, are overtones of an unknown fundamental. Of the various possible values of frequency of the fundamental, what is the largest?

77. Wave-trains of which the wave-lengths are 8 inches and 9 inches, respectively, are component parts of a compound wave-train. Of the various possible values of wave-length of the compound wave-train, what is the shortest?

CHAPTER XIII

FREE VIBRATIONS OF ELASTIC BODIES

131. Vibration of a plucked string (special cases). When a string is pulled to one side and released it vibrates in a manner indicated in Fig. 217. The shape of the string just before it is released is shown in Fig. 217*a*. When the string is released a straight and uniformly downward moving portion develops more and more, as indicated in Fig. 217*b*, until the whole string is moving downwards, as shown in Fig. 217*c*. Then the moving portion grows shorter and shorter, as indicated in Fig. 217*d*, until the whole string is standing still, as shown in Fig. 217*e*. The string then moves upwards through the same series of stages in reverse order until it comes back to its initial condition, as represented in Fig. 217*a*; and so on, over and over again. The sharp bends or corners WW travel to right or left along the string at velocity V, which is given by the equation

$$V = \sqrt{\frac{T}{m}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

where T is the tension of the string in dynes (or "pounds"), m is the mass of one cm. of the string in grams (or of one foot of the string in slugs), and V is the velocity in cm. per second (or feet per sec.). Equation (i) is derived in Art. 133.

When a string is plucked near one end it moves, as indicated in Fig 218. A uniformly moving portion

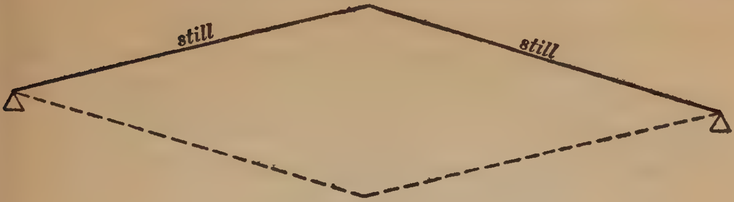

Fig. 217*a*.

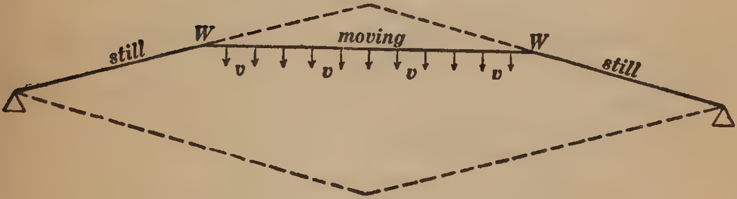

Fig. 217*b*.

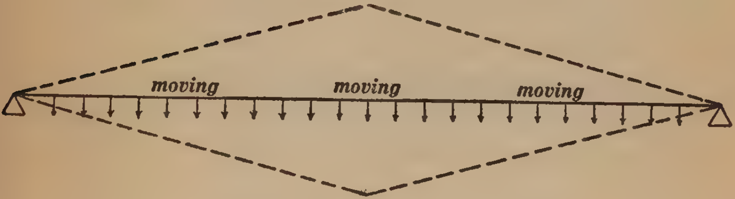

Fig. 217*c*.

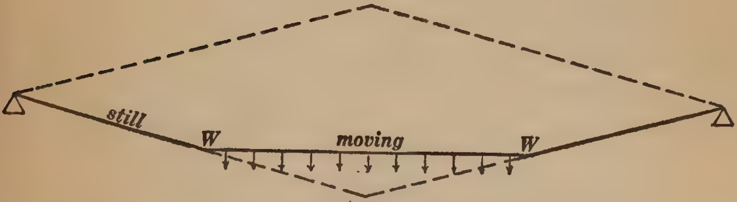

Fig. 217*d*.

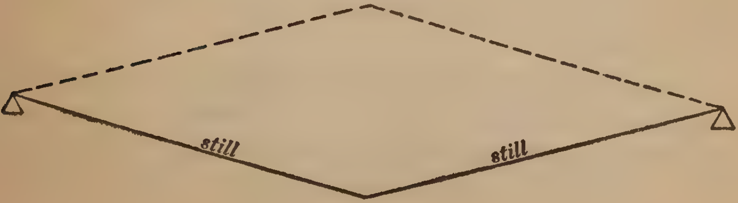

Fig. 217*e*.

develops after the string is released, this moving portion is straight (as in Fig. 217) and equally inclined to the two portions *AB* and *CB*, and as this straight portion travels sidewise it carries the string through a series of configurations from the initial configuration *ABC* to *ADC* and back again, the moving part of the string being straight, as indicated in the figure.

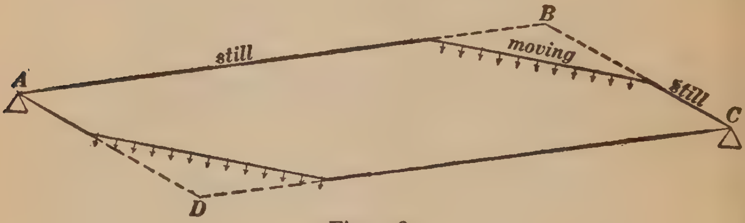

Fig. 218.

If a string is pulled into the shape which is shown by the full line in Fig. 219, and released; two straight moving portions develop, as indicated by the fine dotted lines in the figure, and the middle point *n* of the string never moves.

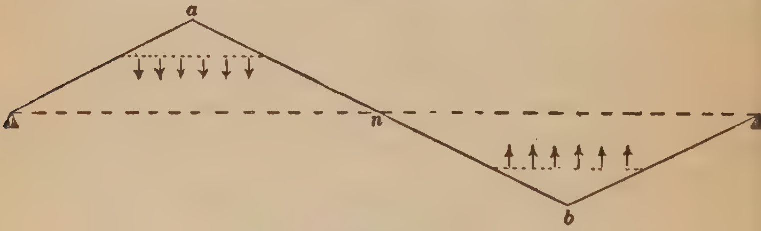

Fig. 219

It is evident from the above discussion that the plucked string of a guitar or mandolin may vibrate in an endless variety of ways, depending upon the initial distortion produced by the operation of plucking. Also the bowed string of the violin and the hammered string of the piano can each vibrate in an endless variety of ways.*

132. Vibration of an air column (special case). Consider a long tube closed at both ends; and imagine the air in one end of the tube to be slightly compressed, and the air in the other end of the tube to be slightly rarefied, as indicated in the upper half of Fig. 220. When the gate valve is suddenly opened the air in the tube surges back and forth along the tube in a manner which is mathematically identical to the motion of a string which is plucked at its center. A uniformly moving body of air at normal atmospheric pressure develops, as indicated in the lower half of Fig. 220, and the successive stages of a complete oscillation may be understood by a proper interpretation of Figs. 217, in which the shape of the

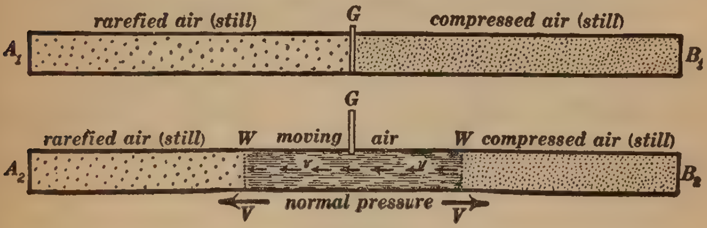

Fig. 220.

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* The motion of a string is discussed mathematically in Chapter VII of *Mechanics*.

A very simple discussion of Fourier's theorem, as applied to a vibrating string, is given in Franklin, MacNutt and 'Charles' *Calculus*, pages 190–209. A very complete discussion of Fourier's theorem is to be found in Byerly's *Fourier's Series and Spherical Harmonics*, Ginn & Co.

A mathematical discussion of the bowed string of the violin is given in an Appendix of Helmholtz's *Tonempfindungen* (translated by Alexander J. Ellis as *Sensations of Tone*, Longmans, Green & Co.).

A beautiful experimental study of the motion of bowed strings is to be found in a paper by Krigar-Menzel and Raps, *Wiedemann's Annalen*, Vol. 44, page 623.

string is to be thought of as a curve whose slope (positive or negative) represents compression or rarefaction of the air at each point of the tube, and whose upward or downward velocity v at any point represents the velocity of the air to right or left along the tube at that particular point.

After the gate valve G in Fig. 220 is opened, the air at the middle of the tube is always at normal atmospheric pressure, and *the air in each half of the tube oscillates as if the tube were cut in two in the middle and each half left open to the air.*

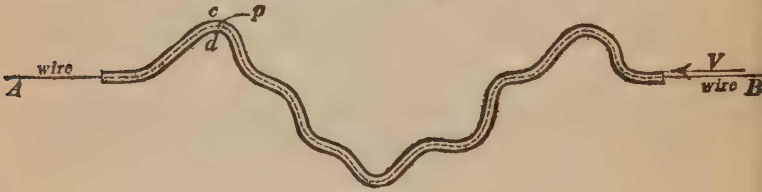

Fig. 221.

133. Velocity of transverse waves on a stretched wire or string. Figure 221 represents a wire which is under a tension of T dynes, and of which the mass of each centimeter is m grams. Imagine the wire to be drawn at velocity V through an irregularly bent tube, and let it be assumed that the wire slides through the tube without friction. Then the wire would not be pulled against the sides of the tube anywhere because of its tension, and it would not be thrown against the sides of the tube anywhere by centrifugal action if

$$V = \sqrt{\frac{T}{m}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Therefore the bend, being once formed on the moving wire, would stand still in space and remain unchanged in shape even if there were no guide tube. Such a *standing* bend on the *moving* wire is a wave traveling along the wire at velocity V.

Proof of equation (i). Tension alone would cause the wire at p in Fig. 221 to push on side d of the tube with a force of T/r dynes per centimeter of the wire, where r is the radius of curvature of the wire at p. Centrifugal action alone would cause the wire at p to push on side c of the tube with a force of mV^2/r dynes per centimeter of the wire. (See Art. 42 of *Mechanics*.) Therefore, if $T/r = mV^2/r$ or if $T = mV^2$ the wire will not push against either side of the tube.

134. Simple modes of vibration. The vibratory motion of a plucked string as described in Art. 131 and the vibratory motion of an air column as described in Art. 132 are extremely simple from the physical point of view, but they are not simple from the mathematical point of view if it is desired to express the motion algebraically; furthermore, when a string or an air column vibrates, as described in Arts. 131 and 132, a particular particle of the string or air *does not* perform harmonic motion. It is desired to describe in simple terms every possible type of oscillation of a string or air column, and *to do this one must think of all the various ways or modes in which a string or air column can vibrate* **with each particle of the string or air column performing simple harmonic motion.** *Any such mode of oscillation is called a* **simple mode.** For example, the string in Fig. 230 is vibrating in three segments, each particle of the string performs simple harmonic motion, and therefore the string is said to vibrate in a simple mode.

The oscillation of a string or air column involves wave motion along the string or air column, and the discussion of what are called simple modes of oscillation can be carried out with a minimum of difficulty by considering the passage of simple (harmonic) wave-trains along the string or air column. This gives rise to the conception of the *standing wave-train*, and a standing wave-train is a simple mode of oscillation under another name.

135. Standing wave-trains. When one end of a stretched rope is moved steadily up and down, the rope soon settles to a steady state of oscillation in which a series of points *n* along the rope remain stationary, while the intervening portions of the rope surge up and down, as indicated in Fig. 222. The heavy line in Fig. 222 shows the position of the rope at a given instant; a snap-shot of the rope, as it were. This oscillatory motion, which is entirely devoid of progressive character, is called a *standing wave-train*. The stationary points *n* are called *nodes*, the intervening portions of the vibrating rope are called *vibrating segments*, and the middle point of a vibrating segment is called an *antinode*. It is important to keep in mind the distinction between an *advancing wave-train* (which is usually called, simply, a wave-train) and a *standing wave-train*. In an advancing wave-train no portion of the rope remains stationary, in fact, every particle of the rope moves in precisely the same way and to exactly the same extent, but not simultaneously, each succeeding particle being a little later in its movements.

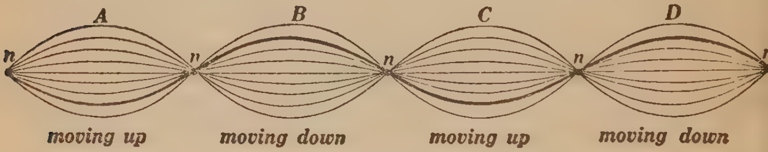

Fig. 222.

In a standing wave-train, on the other hand, the rope does not move at all at the nodes, the amplitude of motion is a maximum at the antinodes, and all portions of the rope move up and down simultaneously, that is to say, all particles of the rope in any vibrating segment move

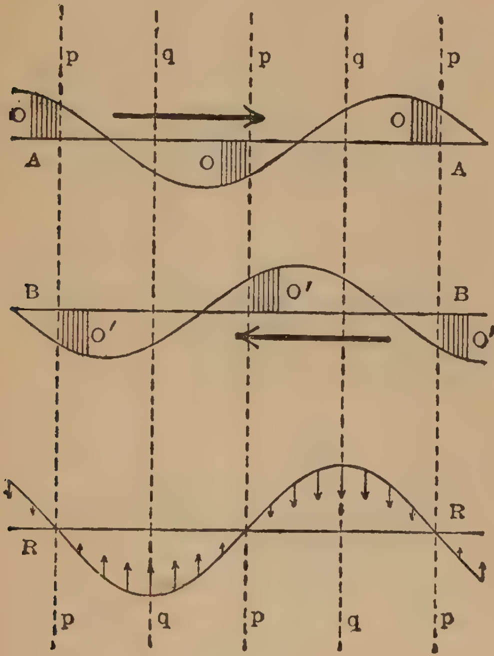

Fig. 223.

up or down together, and the motion is opposite in direction in adjacent segments.

Proposition. *The motion which is represented in Fig. 222 is equivalent to, or may be resolved into, two, oppositely traveling, similar wave-trains.* This proposition is of great importance in the theory of vibrating strings and air columns, and the truth of the proposition may be made evident as follows:

Consider two similar wave-trains *AA* and *BB*, in Figs. 223, 224, 225, and 226, which are traveling in opposite directions as indicated by the heavy arrows. These wave-trains are supposed to be traversing the same portion of the rope at the same time, and *AA* is drawn above *BB* merely to avoid confusion. The actual displacement of each particle of the rope is equal to the sum of the displacements of that particle due to each wave-train, and the actual velocity of each particle is equal to the sum of the velocities of that particle due to each wave-train. Now, the ordinates *O* of the traveling curve

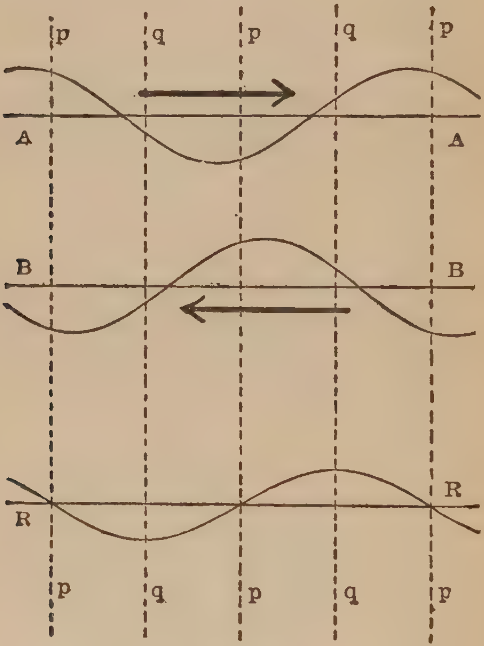

Fig. 224.

AA, Fig. 223, when they reach the point *p* as the wave-train *AA* moves to the right, *are at each instant equal and opposite* to the ordinates *O′* of the traveling curve *BB* when they reach the point *p* as the wave-train *BB* moves to the left. Therefore the points *ppp* of the rope remain stationary.

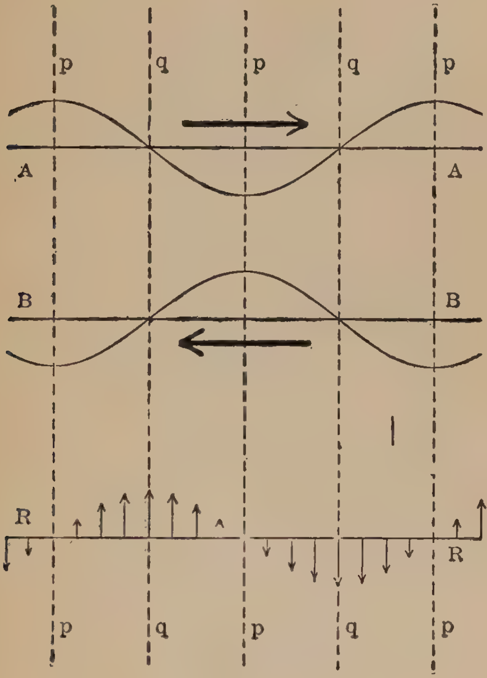

Fig. 225.

The regions between the points *pp*, on the other hand, move up and down (to right and left in case of an air wave in a tube) as the two wave-trains *AA* and *BB* travel through or over each other. The resultant of the two wave-trains *AA* and *BB* is therefore a standing wave-train with nodal points at *ppp*.

Figure 223 shows the positions of the two wave-trains *AA* and *BB* and their resultant *RR* at a given instant. The small vertical arrows show the velocities of the various parts of the rope. Figure 224 shows the positions of the two oppositely moving wave-trains *AA* and *BB* and their resultant *RR* at a later instant when *AA* has moved one-sixteenth of a wave-length to the right and *BB* has moved one-sixteenth of a wave-length to the left. Figure 225 shows the position of *AA* and *BB* and their resultant *RR* (a straight line) at a still later instant when *AA* has moved one-eighth of a wave-length to the right and *BB* has moved one-eighth of a wave-length to the left. The small vertical arrows show the velocities

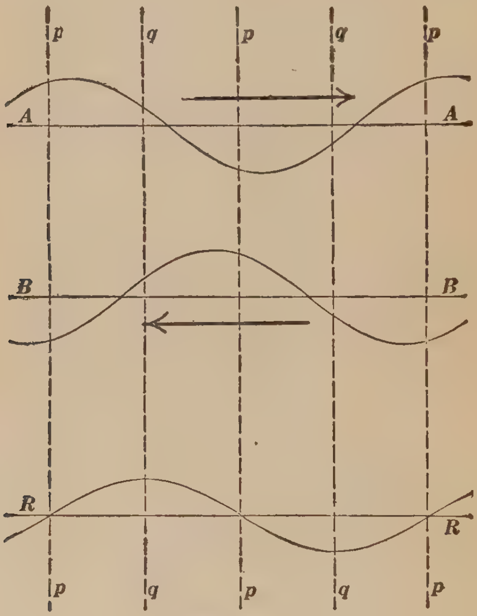

Fig. 226.

of the various parts of the rope as before. Figure 226 shows the position of *AA* and *BB* and their resultant *RR* at a still later instant when *AA* has moved three-sixteenths of a wave-length to the right and *BB* has moved three-sixteenths of a wave-length to the left.

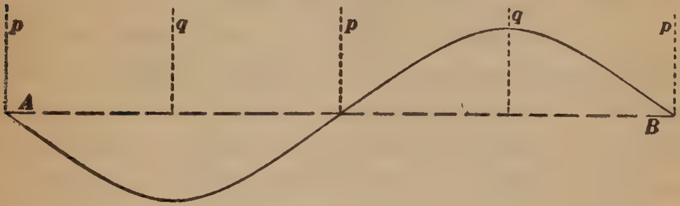

Fig. 227.

Note. The distance from node to node (or from antinode to antinode) in Figs. 223–226 is equal to $\lambda/2$, where λ is the wave-length of each of the advancing wave-trains *AA* and *BB*.

Figure 227 shows the vibrating rope when it is bent to the greatest extent. At this instant the velocity (sidewise) of the rope is everywhere zero. Figure 228 shows the vibrating rope when its sidewise motion is everywhere a maximum. At this instant the rope is straight as indicated by the line *CD*.

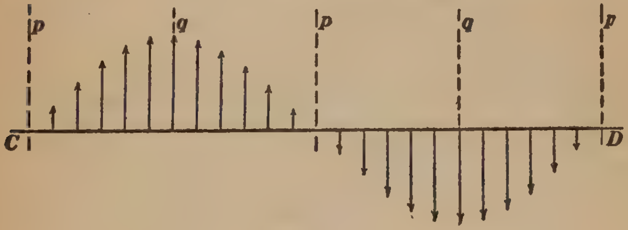

Fig. 228.

(*a*) *The rope never moves at a node of a standing wave-*

train, but is, at times, abnormally stretched. The abnormal stretching at a node p is evident from Fig. 227 when we consider that the normal length of each portion of the stretched rope is the projection of that portion on the straight line AB, so that the rope is most stretched where it is most inclined to this line.

(*b*) *The rope has a maximum of motion at an antinode* q *of a standing wave-train but its amount of stretch is always normal.* The normal amount of stretch of the rope at an antinode is shown by the fact that at an antinode it is always parallel to the line AB, as shown in Fig. 222.

For the case of a stationary or standing wave-train in the air in a tube the above propositions (*a*) and (*b*) become:

(*a*) *The air never moves at a node but it is alternately compressed and rarefied.*

(*b*) *The air at an antinode has a maximum of motion but it is always at normal pressure.*

136. Standing wave-trains by reflection. When a train of waves on a rope (or in an air tube) is reflected from one end of the rope (or air tube), the reflected train and the tail end of the original train travel simultaneously over the same portion of the rope (or air tube), and a stationary wave-train is the result. We will discuss this phenomenon for the case of an air tube for the sake of generality.

If the end of the tube where the reflection takes place is *closed,* then the air at the end of the tube cannot move, but it can, of course, be compressed or rarefied. The reflection must take place so as to satisfy this necessary condition, and therefore a node of the standing wave-train is formed at the closed end of the tube.

If the end of the tube where reflection takes place is *open*, then, of course, the air at the end of the tube can move but it cannot be compressed or rarefied. The reflection must take place so as to satisfy this necessary condition, and therefore an antinode of the standing wave-train is formed at the open end of the tube.

137. Simple modes of vibration of strings. Consider an indefinitely long stretched wire or string, AB, Fig. 229, fixed to a rigid support at end B. Imagine a simple wave-train of wave-length λ to approach the end B and be turned back or reflected. A standing wave-train will be formed as explained in Art. 136, and the nodes of this standing train will be at distances $\lambda/2$ apart, one node being at the rigid end B of the string.

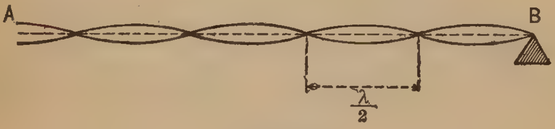

Fig. 229.

This standing wave-train being once established, a rigid support might be placed under the string at any one of the nodes, and the string between this new support and B would continue its vibratory motion unchanged, except, of course, that its motion would be slowly stopped by friction. Therefore, the length l of a vibrating string may be any multiple of $\lambda/2$,* that is,

$$l = \frac{n\lambda}{2} \quad \dots\dots\dots\dots\dots\dots \text{(i)}$$

in which n is *any whole number*. Let V be the velocity with which a wave-train travels along the stretched string, let τ be the period of one oscillation of the string, and let f be the number of oscillations per second (the frequency). Then we have

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* This is merely a brief way of saying that the half-wave-length $\lambda/2$ may be any aliquot part of l, for the length of the string is given in any particular case.

$$\lambda = V\tau \text{ .... (ii)}$$

as explained in Art. 11 of Chapter II. Substituting this value of λ in equation (i) and solving for τ, we have

$$\tau = \frac{2l}{nV} \text{ .... (iii)}$$

or since $f = 1/\tau$, we have

$$f = \frac{nV}{2l} \text{ .... (iv)}$$

This equation expresses the frequency of oscillation of a string (vibrating in a simple mode) in terms of the velocity of transmission V* of waves along the string, and the length l of the string, n being any whole number.

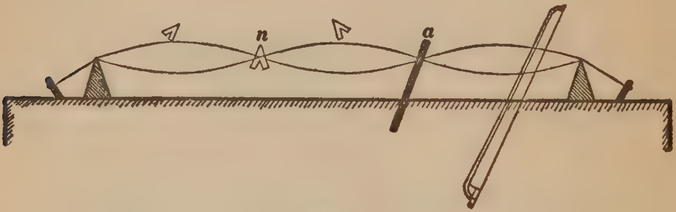

Fig. 230.

When n is unity, the whole string is one vibrating segment, the string vibrates in what is called its *fundamental mode*, and gives what is called its *fundamental tone*. When $n = 2$, the string vibrates in two segments, and gives what is called its *second*† *overtone*. When $n = 3$, the string vibrates in three segments and gives what is called its *third overtone*, etc.

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* See equation (ii) of Art. 133.

† This is really the first overtone, considering that the fundamental is not, properly speaking, an overtone; but it is convenient to designate the order of the overtones by the corresponding values of n.

A string may be made to vibrate approximately* in a fundamental mode by drawing a violin bow across it near one end and touching it with the finger at *a*, as shown in Fig. 230. This figure shows the string vibrating in three segments [$n = 3$ in equation (iv)]. In this case a light paper rider remains on the string at the node *n*, whereas such a rider is thrown off the string at other points as indicated in the figure.

138. Possible modes of vibration of a string. Any type of vibration (free vibration) whatever of a string consists of simultaneous vibrations of the string in its various simple modes. See footnotes to Art. 131.

When a string is plucked, or struck with a hammer, or stroked with a violin bow, it performs simultaneously its various simple modes of oscillation, and gives its fundamental tone together with its various overtones. The relative intensities of the fundamental and various overtones depend upon the position of the point where the string is plucked, or struck, or bowed. Thus a guitar or mandolin string gives a quality of tone which varies quite perceptibly with the location of the point where the string is plucked. In the piano, the best quality of tone is produced when each hammer strikes its string at a point of about one-seventh of the length of the string from one end.

139. Simple modes of vibration of air columns. Waves on wires, such as are described in this text are called *transverse waves*, because the parts of the wire move sidewise as the waves pass by. Waves in air and, in particular, waves in an air tube are called *longitudinal*

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* In Fig. 230 the string vibrates simultaneously in all those simple modes for which a node exists at the point *a*. In the particular case represented in Fig. 230 the 3rd, 6th, 9th, 12th, etc., modes are represented in the vibrations of the string.

waves, because any portion of the air in the tube moves back and forth along the tube (in the line of travel of the waves) as a wave-train passes by. The vertical distance between the curved lines in Fig. 229 represent the extent of sidewise motion of the string at each point, whereas the vertical distance between the curved lines in Figs. 231, 232, and 233 represent the extent of **endwise motion** of the air at each point.

Consider an indefinitely long tube *AB*, Fig. 231, closed at one end *B*. Imagine a simple wave-train of longitudinal waves, of wave-length λ, to approach the closed end *B*, of the tube. This wave-train will be reflected at *B*, the reflected train and advancing train will together form a standing wave-train, and the nodes of this standing wave-train will be at a distance λ/2 from each other, as shown in Fig. 231, the closed end of the tube being a node.

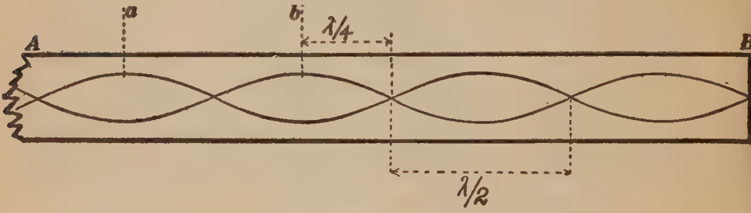


Fig. 231.

Once this standing wave-train is established, an air-tight gate-valve might be placed in the tube at any node, and the air between the gate and the closed end *B* would continue its vibratory motion unchanged, except, of course, its motion would die away because of friction. Therefore

(1). The length *l* of the vibrating air column in a tube which is *closed at both ends* is equal to $n\lambda/2$, where n is any whole number, or the wave-length λ is

$$\lambda = \frac{2l}{n} \quad \text{.....................(i)}$$

where n is **any whole number.**

The air pressure at any antinode in Fig. 231 is invariable and it may be equal to atmospheric pressure, but the air at an antinode surges back and forth along the tube. Such motion is evidently possible at the open end of the tube, and at such an open end the pressure would necessarily be the same as atmospheric pressure.* Consequently if the tube were cut off at any antinode (or at any two antinodes) and the end (or ends) left open, the vibratory motion of the air in the cut-off portion of the tube would continue unchanged except that it would die away rapidly. Therefore we have the following two propositions:

(2). The length l of a vibrating air column in a tube which is *open at both ends* is equal to $n\lambda/2$, or

$$\lambda = \frac{2l}{n} \quad \text{.....................(i)}$$

where n is **any whole number.**

(3). The length l of a vibrating air column in a tube which is *open at one end* and *closed at the other end* is equal to $n'\lambda/4$, or

$$\lambda = \frac{4l}{n'} \quad \text{.....................(ii)}$$

where n' is **any odd number.**

Therefore, proceeding exactly as in Art. 137, we get

$$f = \frac{nV}{2l} \quad \text{...................(iii)}$$

---

* This is only an approximate statement, but it must suffice here.

where f is the frequency of vibration of an air column which is closed at both ends or open at both ends, and n is any whole number, and

$$f' = \frac{n'V}{4l} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

where f' is the frequency of vibration of an air column which is closed at one end, and n' is any odd number.

In all of the above equations l is the length of the air column and V is the velocity of sound in air (see Art. 9 of Chapter I).

A very beautiful experiment showing the vibration of an air column in various simple modes is the following:

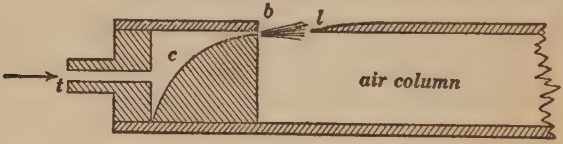


Fig. 232.

An ordinary whistle mouthpiece is fixed to the end of a long glass tube. A weak blast of air causes the whistle to sound a low-pitch tone, and with increasing air pressure the air column in the tube breaks up into a greater and greater number of vibrating segments, giving a higher and higher tone. These vibrating segments may be rendered visible by placing lycopodium powder in the tube and maintaining a sufficiently constant air pressure to give an invariable tone on the whistle. The air, as it surges back and forth (parallel to the length of the tube) in the vibrating segments, sweeps the lycopodium powder into small heaps at the nodes. It is best in this experiment to have the end of the tube closed to prevent violent air currents, and to use dry air in order that the lycopodium may not become moist and adhere to the walls of the tube.

140. The organ pipe is a device, very similar to the bark whistles known to children, in which a column of air is set into vibration and caused to give a musical tone. Sectional views of complete organ pipes are shown in Figs. 233 and 234, but Fig. 232 shows the details of construction and action. Air under moderate pressure enters at *t*, flows through the chamber *c*, issues as an air blast *b*, and strikes against the sharp lip *l*. The quavering of this air blast starts the air column vibrating feebly, these vibrations react upon the air blast and cause it to play from one side to the other of the lip *l*, and this effect reinforces the vibrations so that they quickly become energetic, and they are sustained as long as the air blast continues.

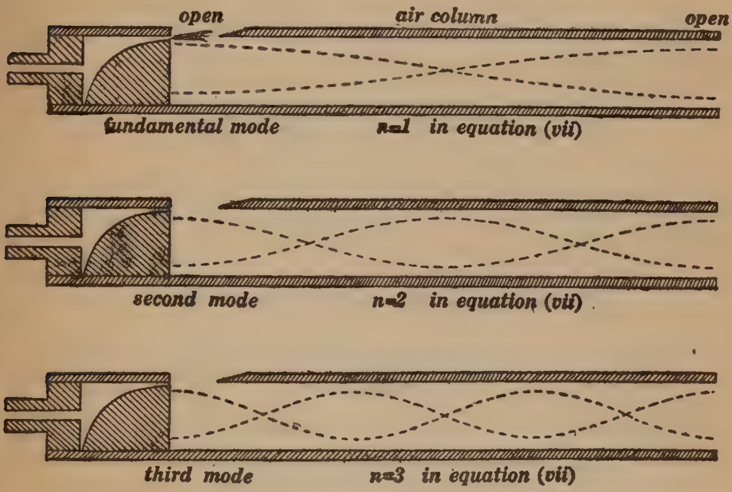
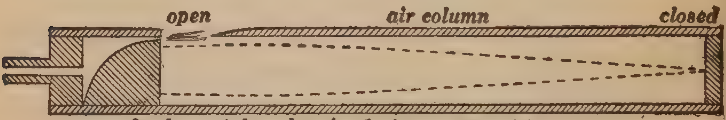


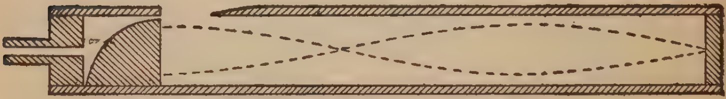
Fig. 233.

The dotted curves in Fig. 233 are intended to show the character of the motion of the air in an organ pipe which is open at both ends, and the dotted curves in Fig. 234 are intended to show the character of the motion of the air in an organ pipe which is closed at one end.

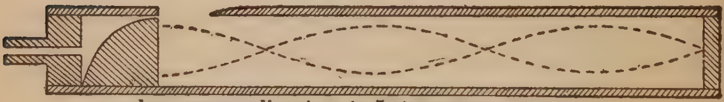
Figures 233 and 234 represent simple modes or oscillation of organ pipes. As a matter of fact, however, an organ pipe always performs its various simple modes of oscillation simultaneously, and the musical tone produced by an organ pipe is therefore composed of a series of simple tones blended together. With broad short pipes the overtones are usually very weak, and the sound approaches what is called a simple tone in character. With long narrow pipes, the overtones are more pronounced; and when a long narrow organ pipe is blown strongly a single one of its overtones is usually produced, the others remaining inaudible.



fundamental mode $n' = 1$ *in equation* (x)



mode corresponding to $n' = 3$ *in equation* (x)



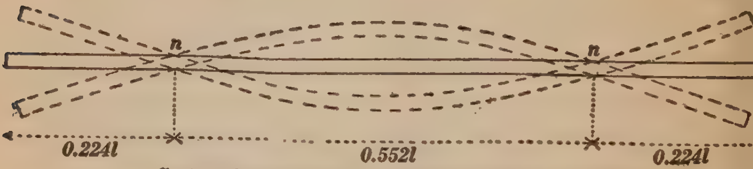
mode corresponding to $n' = 5$ *in equation* (x)

Fig. 234.

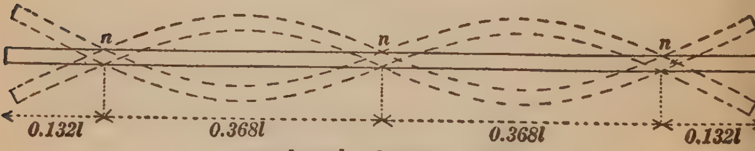
The clarinet, the flute, the cornet, and the bugle are examples of musical instruments in which tones are produced by the vibration of air columns. In the clarinet and the flute, the length (and thereby the pitch) of the vibrating air column is altered by uncovering openings in the side of the tube. In the cornet, the length of the vibrating air column is altered by including or excluding extra lengths of tube by means of valve connections. In the bugle, the length of the vibrating air column is fixed, and the various notes are obtained by causing the air column to vibrate in one or another simple mode; therefore the only tones which can be produced on a bugle are the tones whose vibration frequencies are proportional to the numbers 1, 2, 3, 4, 5, 6, 7, 8, etc. In fact, the tones 3, 4, 5, and 6 are the ones ordinarily employed in the bugle.

141. Vibrations of rods. The longitudinal vibrations of rods are exactly similar to the longitudinal vibrations of air columns and need not be discussed. When a rod is struck on one side with a hammer, it is caused to vibrate with a *sidewise* or *transverse* motion. One can always hear a number of distinct tones in the sound produced by a vibrating rod when the vibrations have been produced by a hammer blow, and each of these tones is due to a *simple mode* of vibration of the rod; a simple mode of oscillation being a type of oscillation in which every particle of the rod performs simple harmonic motion of a certain frequency.

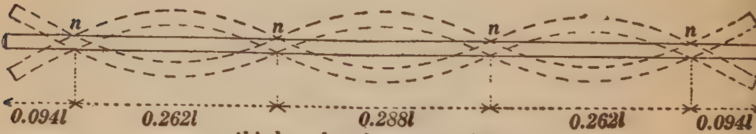
In the case of vibrating strings and air columns, the vibration frequencies which correspond to the various simple modes of oscillation are approximately in proportion to the successive whole numbers. In the case of rods, however, the vibration frequencies which correspond to the various simple modes of transverse oscillation do not form a simple series of numbers. Thus, Fig. 235 represents three successive simple modes of oscillation of a long steel rod.* The frequencies of the successive simple modes are given in terms of the frequency of the fundamental mode which is taken arbitrarily as 1000. The fundamental mode has two nodes, the second mode has three nodes, and the third mode has four nodes. The locations of the nodes are indicated in terms of the length *l* of the rod. In order to cause a rod to vibrate according to one of the sketches in Fig. 235, the rod would have to be supported at one or the other set of nodes and excited by a violin bow. The positions of the nodes and the relative frequencies of the various



first or fundamental mode; frequency taken as 1000



second mode; frequency 2754



third mode; frequency 5402

Fig. 235.

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* This figure is based on a theoretical paper by Seebeck which was published in 1848. See Winkelmann's *Handbuch der Physik.*

modes depend to some extent upon the ratio of length to diameter of rod.

The *tuning fork* is a stiff rod bent into the form shown in Fig. 236. The character of its fundamental mode of vibration is shown by the dotted lines. The two nodes *nn* are near together, and the intervening segment, together with the metal post *P*, move up and down through a small amplitude and cause the sounding board upon which the fork is mounted to vibrate in unison with the fork. The second simple mode of vibration of a fork usually gives a tone two or three octaves above the fundamental, and it dies out quickly after the fork is set into vibration by a hammer blow, leaving the fundamental alone. The tuning fork, therefore, gives a simple tone.

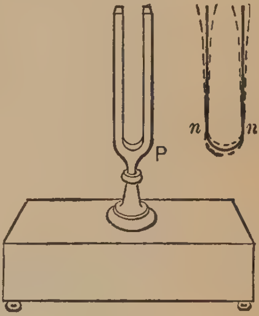

Fig. 236.

142. Vibrations of plates. When a plate, such as a circular saw of steel, is struck on one side with a hammer, it is set into vibration, and one can always hear a number of distinct tones in the sound produced. Each of these tones corresponds to a simple mode of vibration of the plate, a simple mode of oscillation being a type of oscillation in which every particle of the plate performs simple harmonic motion of a certain frequency. The vibration frequencies which correspond to the various simple modes of oscillation of a plate do not form a simple series of whole numbers.

An elastic plate may be made to perform a simple mode of oscillation by supporting it in a clamp as shown in Fig. 237, touching the fingers lightly against the plate

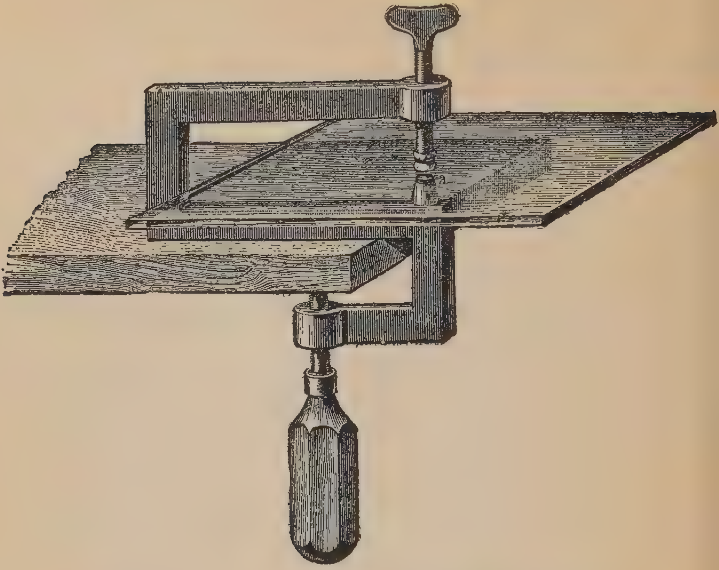

Fig. 237.

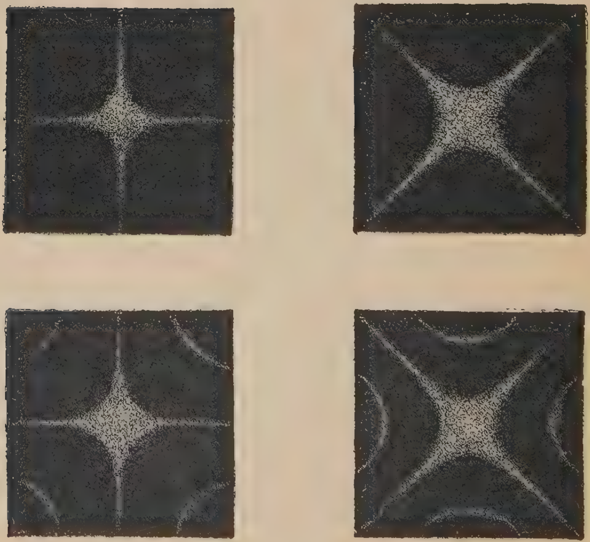

Fig. 238.

in the positions (found by trial) corresponding to the nodes, and drawing a violin bow across the edge of the plate. When sand is strewn upon the plate, it is thrown away from the vibrating segments and heaped along the nodal lines. Thus Fig. 238 shows four sand figures obtained in this way. These figures were first studied by the Italian physicist Chladni and described in his treatise on acoustics in 1787.

The bell may be looked upon as a cup-shaped plate. When a bell is struck, a number of distinct tones may be heard. These tones correspond to simple modes of vibration of the bell, and their frequencies are not usually in proportion to the whole numbers 1, 2, 3, 4, 5, etc. Bell founders have, however, learned to produce bells of such shape as to make the frequencies of the more prominent overtones have simple integral ratios, thus giving very pleasing tones.

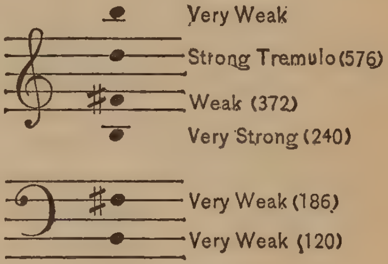

Fig. 239.

Figure 239 shows the more prominent tones of a Russian bell in the library of Cornell University. The numbers represent the number of vibrations per second. The tone marked "tremulo" consists of *two tones* differing very slightly in pitch, but of nearly the same loudness, and the tremulous character is produced by the beats of the tones.

LEADING QUESTIONS

L. Q. 93. Plot a curve of which the abscissas represent elapsed times and of which the ordinates represent the varying displacement (up or down) of a point *p* on the string in Fig. 217: (*a*) When the point *p* is at the middle of the string, and (*b*) When the point *p* is half-way between the middle and one end of the string.

L. Q. 94. Plot a curve of which the abscissas represent elapsed times and of which the ordinates represent the varying sidewise velocity (up or down) of a point *p* of the string in Fig 217: (*a*) When the point *p* is at the middle of the string, and (*b*) When the point *p* is half-way between the middle and one end of the string.

L. Q. 95. Taking T as the tension of the string in the tube in Fig. 221, find the expression for the sidewise push of unit length of the string against the side of the tube when the string is stationary. Taking m as the mass of the string per unit length, find an expression for the sidewise push of unit length of the string against the side of the tube when the string has no tension and is sliding through the tube at velocity v. Neglect friction of the sliding string and consider a place where the tube is the arc of a circle of radius R.

L. Q. 96. Precisely what is meant by a simple mode of vibration of a string or air column?

L. Q. 97. Define the terms node, antinode and vibrating segment as applied to a string or air column vibrating in a simple node.

L. Q. 98. What change takes place at a node of a standing wave-train on a string? What change at a node of a standing wave-train in the air in a tube? What change takes place at an antinode in each case of standing waves?

L. Q. 99. What does the vertical distance between the dotted lines in Figs. 233 and 234 represent?

PROBLEMS

78. A rope 10 meters long has a mass of 2·5 kilograms, the rope is stretched between two rigid supports, and its tension is equal to the weight of 100 kilograms (98,000,000 dynes). Find the frequency of the fundamental mode of transverse vibration of the rope. Find the frequency of the mode for which the rope has three vibrating segments.

79. If the center of the rope of problem 78 is pulled 50 cm. to one side and released, how fast will the central part of the rope move sidewise after release? How long will it take the rope to change from the initial condition (see Fig. 217*a*) to the condition shown in Fig. 217*c*?

80. A guitar string is 60 cm. long. Find how much the string must be shortened (by pushing it down against a fret) to increase its frequency in the ratio 8 to 9, neglecting the slight increase of tension.

81. Air waves in a tube travel at the normal velocity of sound in air (see Art. 9 of Chapter I). Calculate the lowest frequency of oscillation of the air in a tube 3 meters long: (*a*) When the tube is open at both ends, and (*b*) When the tube is open at one end and closed at the other end.

82. Calculate the next-to-the-lowest frequencies of oscillation in problem 81.

83. Sound travels 3·8 times as fast in hydrogen as in air (temperature being the same). Calculate the frequencies of oscillation of the tube in problems 81 and 82 when the tube is filled with hydrogen.

84. A cornet has a tube 4 feet long. Calculate the length of the auxiliary tube required to lower the frequency of the cornet tone in the ratio of 9 to 8. The air column in a cornet vibrates as in an organ pipe open at both ends.

CHAPTER XIV

FORCED VIBRATIONS AND RESONANCE

143. Free vibrations and forced vibrations. The vibrations which a body performs when it is disturbed in any way and left to itself are called its *free* or *proper* vibrations. When a simple wave-train of sound waves of any wave-length strikes a body, the body is made to vibrate in unison, or in the same rhythm, with the impinging waves.* Such vibrations are called *forced* or *impressed* vibrations. A compound wave-train causes a body to perform simultaneously the various simple vibrations which correspond to the various simple wave-trains which enter into the composition of the compound wave-train.

144. Damping. Vibrations are said to be *damped* when they die out quickly. This damping effect is due in part to the dissipation of energy in the body in the form of heat, as it is repeatedly distorted, and in part to the giving up of energy to the surrounding air. Thus, the vibrations of a light body which exposes considerable surface to the air, a diaphragm, for example, die out quickly: whereas, the vibrations of a heavy and highly elastic body, like a tuning fork, are but slightly damped. A heavy tuning fork performs several thousand perceptible vibrations when struck. The column of air in an organ pipe performs several hundred perceptible variations after the exciting cause ceases. A drum-head performs very few perceptible vibrations when struck.

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* This statement refers to the ultimate character of the motion and not to the motion which takes place during the time that the steady state of vibration is being established. During the time that the steady state of vibration is being established, the body behaves as if its free vibrations and the ultimate forced vibrations existed simultaneously, and, as the free vibrations die out, the forced vibrations are left alone.

145. Resonance. Very perceptible vibrations are impressed upon a light diaphragm by sound wave-trains of any wave-length (frequency), but *the vibrations which are impressed upon a heavy body, of which the damping is slight, are much more violent when the impressed frequency approaches the frequency of the free vibrations of the body.* Thus, the sound of a tuning fork (removed from its sounding board) is perceptibly reinforced when it is held near the open end of a tube of any length; but if the length of the air column is adjusted, by pouring water into the tube for example, the sound becomes louder as the proper frequency of the air column approaches that of the fork; and it reaches a very distinct maximum of loudness when the impressed vibrations become proper to the air column.

When the vibrations of a body are subject to very slight damping, the impressed vibrations at proper frequency are very energetic as compared with the impressed vibrations of improper frequency. Thus, a massive tuning fork, mounted upon its sounding board, is thrown into very energetic vibration by a tone of its proper frequency sustained for four or five seconds, but it is scarcely affected at all by a tone of which the frequency differs very slightly from the proper frequency of the fork. *This pronounced maximum violence of impressed vibration at proper frequency is called* **resonance,** *and the vibrating body or air column is called a* **resonator.**

When impressed vibrations are proper to a body, the impulse of each successive wave adds to the existing motion, and the vibrations increase in violence until the energy given to the vibrating body by the impinging waves is all dissipated by damping. It is for this reason that the impressed vibrations become quite energetic when the damping is small. When, however, improper vibrations are impressed upon a body, the periodic forces with which the waves act upon the body must take the place, more or less, of the *internal elastic forces* which ordinarily cause the body to vibrate; consequently the vibrations cannot become very energetic.

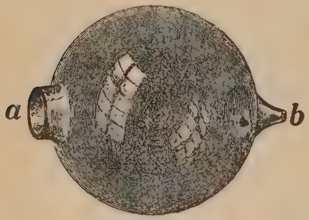

Fig. 240.

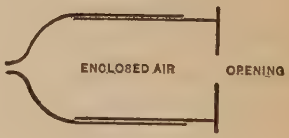

Fig. 241.

146. Analysis of compound tones by means of resonators. A resonator of which the proper frequency coincides with the frequency of one of the component tones of a compound tone or clang, responds to the component tone by resonance and increases its loudness. Thus, if the frequencies of the fundamental and successive overtones of a string are 100, 200, 300, 400, etc., vibrations per second, a resonator of which the proper frequency is 400 vibrations per second will be set into energetic vibration by the sound of the string, and this particular overtone will be increased in loudness.

Any component tone of a clang may be easily detected, or brought to notice, by intermittently strengthening it by means of a resonator tuned to unison with it. A convenient resonator for this purpose is shown in Fig. 240. It consists of a hollow glass ball with an open mouth at *a* and an open tip *b*. The open tip *b* is inserted into the ear, and the sound waves act upon the inclosed air through the opening *a*. In order to analyze a clang, one after another of a series of such resonators must be applied to the ear. Figure 241 shows an adjustible resonator due to Koenig. It consists of two brass cups, one sliding into the other, and the outer cup ends in a small nipple which is inserted into the observer's ear.

147. Vowel sounds. Every vowel sound is characterized by one or two *tones of* **definite** *pitch.* The characteristic tones of some of the vowels, as determined by Helmholtz,* are shown in the following table, and Fig. 242 gives these characteristic tones in terms of the ordinary musical notation.

Characteristic Tones of Vowel Sounds.

Vowel.	Tone.	Vibration Frequency.†
ṳ̄ as in *rude.*	f	173
ō as in no.	c''	517
ā as in *paw.*	g''	775
ä as in *part.*	$d^{\flat'''}$	1,096
ā as in *pay.*	f and b'''	346 and 1,843
ĕ as in *pet.*	c'''''	2,068
ē as in *see.*	f and d''''	173 and 2,322

In producing a given vowel sound, the mouth cavity is shaped so that the contained air has a proper frequency of vibration which corresponds to the characteristic tone of the vowel to be produced. The sound from the vocal chords is very complex (it contains tones of almost every pitch), and that particular tone which is in unison with the free or proper vibrations of the air in the mouth cavity, is greatly strengthened by resonance, thus producing the desired vowel sound.

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* See *Sensations of Tone* (a translation of Helmholtz's *Tonempfindungen*, by A. J. Ellis), pages 153–173.

A very complete study of vowel sounds has been made by D. C. Miller. See pages 215–251, *The Science of Musical Sounds*, by D. C. Miller, The Macmillan Co., New York, 1916.

† Complete vibrations per second.

In ordinary speech, the sound produced by the vocal chords is very harsh and suffices to excite the resonance of the mouth cavity for the production of any vowel sound. The smooth tone of a singer, however, may not contain the characteristic tone of the vowel, so that this tone cannot be strengthened by resonance. In this case, those overtones of the note which is sung, which are nearest the characteristic tone of the vowel for which the mouth cavity is set, are slightly strengthened, and in this way the vowel sound is produced, although rather incompletely. It is a well-known fact that spoken words are much more easily understood than words which are sung, and the difference in distinctness is largely due to the imperfect character of the vowels when sung.

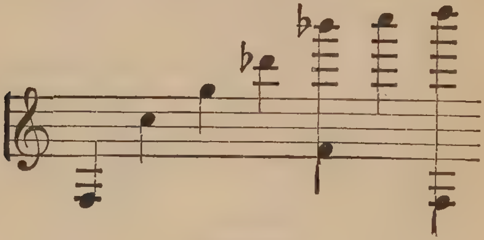

Fig. 242.

Overtones near a given pitch are more widely separated in a note of high pitch than in a note of low pitch, so that the mouth cavity, shaped to give the characteristic tone of a vowel, is less likely to produce the desired effect with high notes than with low notes. Thus, the words of a soprano singer are less distinct than the words of a bass singer of the same schooling.

The proper tones of the mouth cavity when it is shaped for the production of the vowel ō, a and ä may be easily heard by thumping against the cheek when the mouth is prepared to sound these vowels. A very striking experiment is the following: The lungs are filled with purified hydrogen, and a series of words like *rude*, *no*, *paw*, *part*, *pay*, *pet*, *see*, are spoken very deliberately and carefully. The mouth cavity is automatically shaped to produce the various characterizing tones, but the presence of hydrogen instead of air in the mouth raises the pitch of the mouth cavity nearly two octaves and the result is that the attempt to produce the vowels ṳ̄, ō, ā̤, ä, ā, etc., fails in a most laughable manner.

148. The phonograph. A thin diaphragm carries a light tool which scratches a minute groove in a smooth rotating cylinder made of a hard wax-like compound. A sound striking the diaphragm impresses vibrations upon it and causes the tool to cut a groove of varying depth. A record of the sound is thus made upon the cylinder. To reproduce the sound, a round-ended tool which is attached to the diaphragm is adjusted to follow the groove, and the varying depth of the grove lifts the tool up and down, causing the diaphragm to vibrate as before and the sound is reproduced. This description applies to the old cylinder type of phonograph, but the action of the disk type of phonograph is not essentially different.

LEADING QUESTIONS

L. Q. 100. A heavy weight is fastened to one end of a flat spring and the other end of the spring is clamped in a vise. When the weight is pulled to one side and released it makes 4 complete vibrations per second. These are called the *free* vibrations of the weight, and 4 vibrations per second is called the *proper frequency* of the weight.

(*a*) An alternating force of very high frequency n is exerted on the weight, causing the weight to oscillate at high frequency n. Is the applied force used chiefly to accelerate and decelerate the weight or to bend the spring? Why?

(*b*) An alternating force of very low frequency n' is exerted on the weight, causing the weight to oscillate at frequency n'. Is the applied force used chiefly to accelerate and decelerate the weight or to bend the spring? Why?

(*c*) A very small alternating force of proper frequency is applied to the weight, and after a while the weight comes to a steady state of oscillation at its proper frequency. When this steady state of vibration is reached the applied force overcomes friction only, it does not have anything to do with the accelerating or decelerating of the weight or the bending of the spring. Explain why.

(*d*) Explain why the amplitude of the vibrations is very small in *a* and *b* as compared with *c*.

L. Q. 101. A bass voice sings the vowel *a* as in *father* on a note of which the frequency is 100 vibrations per second. A soprano sings the same vowel on a note of which the frequency is 700 vibrations per second. Make a diagram in which the overtones of both notes are laid down as points at distances proportional to their frequencies, and make a point in this diagram showing the frequency of the tone which characterizes the vowel. Which of the two singers produce the vowel most distinctly and why?

L. Q. 102. When the lungs are filled with a mixture of 7 parts of hydrogen to 6 parts of air by volume, the vowel *a* as in *part* is produced when the mouth cavity is set to produce *a* as in *paw*. Explain.

L. Q. 103. What is the effect on vowel sounds when a phonograph record is run at abnormally high speed?

PROBLEM

85. A vibrating tuning fork is held over the mouth of a tall glass jar, and water is poured into the jar slowly, thus shortening the air column in the jar, and when the length of the air column has been reduced to 35 centimeters, the air column is set into vibration by the fork by resonance. If this is the shortest air column which will respond in this way to the tuning fork, what is the frequency of vibration of the fork? What is the next longer length of air column which will respond to the fork?

CHAPTER XV

THE EAR AND HEARING

149. The human ear. It was pointed out by Helmholtz that our perception of the various simple tones in a clang must depend upon the existence of a series of organs (the end organs of the auditory nerves) in the ear, each of which has a proper vibration frequency and is sensitive (by resonance) to simple tones nearly in unison with it. Thus, a compound tone excites those particular elements in the ear whose vibration frequencies correspond to the various simple tones of the clang. This action may be illustrated by means of the piano as follows: A musical tone of characteristic quality, a vowel sound, for example, is sung loudly against the sounding board of a piano of which the dampers are raised so as to leave the strings free. Those strings which are capable of vibrating in unison with the various simple tones of the clang are set into vibration by resonance, and one can hear a continuation by the piano of the vowel sound after the singing ceases. Imagine each string of a piano to be connected to a nerve fiber, and we have an apparatus which would perceive tones as they are actually perceived by the ear.

The elements in the ear which respond to tones by resonance are supposed to be the shreds of the *basilar membrane* which are stretched across a long slender cavity called the *cochlea.* This cavity is coiled upon itself like a small shell,* hence its name.

150. Persistence of sound sensations. When the stimulation of a nerve ceases, the accompanying sensation continues for a length of time which depends upon the intensity of the stimulation and which varies greatly with the different nerves. Sensations of light persist much longer than sensations of sound. Thus, an intermittent light gives a sensibly continuous sensation when the flashes follow one another at the rate of thirty or forty per second, whereas an intermittent sound, for example, the sound of a tuning fork of high pitch which is shut off from the ear periodically has been found by Mayer to give a continuous sensation when the frequency of intermittence reaches 135 per second.

The persistence of sensations of light are shown by the fact that a rapidly moving spark looks like a streak of light, and the so-called *stroboscopic* effect depends upon the persistence of vision. Thus, if a rotating wheel is illuminated by a series of quick flashes of light the wheel either appears to stand still, or rotate slowly forwards, or rotate slowly backwards according as the time interval between the flashes is equal to, or slightly greater, or slightly less than the time for one spoke of the wheel to move forwards to the position of the next spoke.

151. Consonance and dissonance.† An intermittent or fluctuating tone produces an unpleasant sensation which is called *discord* or *dissonance.* A steady tone, or one which fluctuates so rapidly as to give a steady sensation, produces a pleasing effect which is called *concord* or *consonance*. These terms discord (or dissonance) and concord (or consonance) are used ordinarily to describe the effects produced by two or more simultaneous tones, that is, they are used to describe the relations of tones in music. Nevertheless, the above definitions are physically correct.

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* See *Sensations of Tone* (Ellis' translation of Helmholtz's *Tonempfindungen*), pages 188–226.

† The theory of consonance and dissonance as here developed is used by Helmholtz in his very interesting theory of music. This theory is given in simple form in Chapter XVI.

Consider a tone which is intermittently shut off from the ear, the intermittence beginning at low frequency and increasing to greater and greater frequency. The dissonance at first increases with increasing frequency of intermittence, reaches a maximum, then falls off and disappears entirely when the frequency of intermittence becomes so great as to give a continuous sensation of tone. The frequency of intermittence for which the dissonance is a maximum, and the frequency of intermittence for which the sensation becomes continuous, depend upon the vibration frequency of the tone which is used, as shown in the accompanying table, which is from experiments made by Mayer.

Vibration frequency of intermittent tone.	Frequency of intermittence.	
	When tone becomes smooth.	When discord is a maximum
64	16	6·4
128	26	10·4
256	47	18·8
384	60	24·0
512	78	31·2
640	90	36·0
768	109	43·6
1,024	135	54·0

The fluctuations which produce dissonance in music are the fluctuations due to beats, which are described in the following paragraph. When two tones are in unison they give a smooth sensation. If the vibration frequency of one tone is slowly increased, beats occur with greater and greater frequency, the resulting intermittent sound produces a more and more discordant sensation which soon reaches a maximum, after which the discord decreases and finally disappears when the beats reach a sufficiently high frequency.

152. Beats and combination tones. Consider two simple tones of which the vibration frequencies, f and f', are nearly the same. At a certain instant the wave-trains which constitute these two tones will be in like phase as they enter the ear, giving a maximum loudness of sensation. When the tone of higher pitch has gained half a vibration (or half a wave-length) over the other, the wave-trains will be opposite in phase as they enter the ear, giving a minimum loudness of sensation. When the higher tone has gained a whole vibration over the other the waves will again enter the ear in like phase, and the loudness will again be a maximum, and so on. These periodic changes of loudness of two tones of approximately the same frequency constitute what are called *beats*. The number of beats occurring in one second is equal to $f - f'$.

Very prominent beats may be produced by sounding two similar organ pipes simultaneously, the open end of one of the pipes being partly covered to make it give a tone of slightly lower pitch than the other pipe.

Combination tones. The principle of superposition, namely, that two trains of waves may pass through the same region at the same time without affecting each other, or that an elastic system may perform a series of simple harmonic movements simultaneously without the various vibratory movements being affected by each other is not strictly true. The failure of this principle of superposition is due to what may be termed the incomplete elasticity of ordinary substances, that is, to the fact that the elastic forces brought into play by the sum of the two distortions is not exactly equal to the sum of the elastic forces brought into play by the two distortions separately. The principle of superposition fails most decidedly when an imperfectly elastic substance is vibrating very violently.

Two weak tones do not sensibly affect each other when they are transmitted simultaneously through the same region of air or when they act simultaneously upon any elastic or approximately elastic system, such as the transmitting mechanism of the ear. As the tones grow louder, however, certain other tones produced by their mutual action begin to be heard. These accompanying tones are called *secondary tones* or *combination tones*. The imperfectly elastic chain of bones in the ear has most to do with the formation of combination tones. Everyone, perhaps, has noticed the harsh rattle in the ear which accompanies the loud sound of a dinner bell. This is due to the incomplete elasticity of the tissues which connect the tiny bones together. Indeed, it is a sort of rattling effect of the bones on each other, and it is the same thing essentially as a combination tone, only greatly exaggerated.

Difference tones. The most prominent combination tone is that of which the frequency is equal to the difference of the frequencies of the two primary tones. This is called the *difference tone of the first order*. This difference tone forms difference tones with each of the primary tones and these are called *difference tones of the second order*, and so on.

Summation tones. Very much less prominent than the difference tones is the combination tone of which the frequency is equal to the sum of the frequencies of the two primary tones. This is called the *summation tone of the first order*. Summation tones of the first order form summation tones with each primary tone and these are called *summation tones of the second order*, and so on.

Example. The easiest method, perhaps, to produce an audible combination tone is to use two large steel bars of which the vibration frequencies are, say, 500 and 600 vibrations per second, respectively. When these bars are properly suspended (at the nodes of the fundamental mode, see Fig. 235), and struck in quick succession with a heavy hammer, a very distinct tone is heard, of which the pitch is 100 vibrations per second.

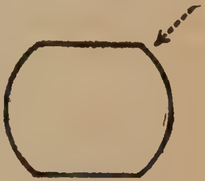

Fig. 243.

A single steel bar, of which the section is of the shape shown in Fig. 243, gives a tone of low pitch when struck on one of the flat faces, and a tone of high pitch when struck in a direction at right angles to this. A hammer blow in the direction of the dotted arrow causes the bar to give both tones simultaneously, and the first-order difference tone may be heard distinctly throughout a large room.

CHAPTER XVI

THE PHYSICAL THEORY OF MUSIC*

153. Pitch intervals and their measurement. Consider two musical tones of different pitch (different frequencies of vibration). The pitch interval between these tones is usually expressed in terms of the ratio of the vibration frequencies, and any two musical tones which have the same ratio of vibration frequencies are said to have the same pitch interval between them. Thus the notes *c* and *g* on the piano have the same frequency ratio 2 : 3 in every octave of the entire piano scale; in one case it may be 200 : 300, in another case, 400 : 600, and so on; and the pitch interval between *c* and *g* is the same in all octaves.

Pitch intervals are ordinarily expressed in terms of frequency ratios. In the discussion of the tempered scale, however, it is more convenient to express pitch intervals in terms of the logarithms of their frequency ratios. Consider, for example, a number of tones of which the vibration frequencies are n, an, a^2n, a^3n, etc. The frequency ratio of any two successive tones in this series is a, and therefore the pitch intervals are all equal. Let p represent the value of this pitch interval. The frequency ratio of the first and third tones is a^2, and their pitch interval should be expressed as $2p$; the frequency ratio of the first and fourth tone is a^3, and their pitch interval should be expressed as $3p$, and so on. In order that we may add together the two numbers which express two pitch intervals to get the number which expresses the sum of the two pitch intervals, we must use for p the logarithm of the frequency ratio as may be seen from the following tabular arrangement.

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* The physical theory of music has been developed mainly by Helmholtz, and it is given in his great work entitled *Tonemfindungen* (translated by Alexander J. Ellis, title of translation, *Sensations of Tone*). A very interesting discussion of harmony in music is Helmholtz's Popular Lecture entitled *On the Physiological Causes of Harmony in Music*, translated by A. J. Ellis, *Helmholtz's Popular Lectures*, First series, Longmans, Green & Company, 1873. A very interesting and important book is D. C. Miller's *The Science of Musical Sounds*, The Macmillan Co., New York, 1916.

Frequency ratios,	a	a^2	a^3	a^4	etc.
Logarithms of frequency ratios,	l	$2l$	$3l$	$4l$	etc.
Pitch intervals,	p	$2p$	$3p$	$4p$	etc.

154. Consonance and dissonance of compound tones. The consonance and dissonance of simple tones have been discussed in Art. 151. Tones ordinarily used in music are, however, compound, and musical consonance and dissonance depends to a great extent upon this fact. The fundamental tone usually predominates in a compound tone, and the overtones 2, 3, 4, 5, 6, and 8 usually occur, decreasing in loudness in the order given. *The following discussion of consonance and dissonance is limited to the influence of these six overtones.*

When two compound tones A and B are in unison, their respective overtones are in unison also; the combined sound of the two tones is therefore entirely free from roughness due to beats; and the two tones are *completely consonant.*

When two compound tones A and B are *not* in unison, then, even if their pitch difference is so great that their fundamental tones do not produce audible beats, some of the overtones of A will generally be near enough to some of the overtones of B to produce audible beats and give distinct dissonance. If the pitch of B is slowly raised or lowered starting from unison with A, this dissonance passes through a very marked minimum value every time one (or more) of the overtones of A comes into unison with one (or more) of the overtones of B. The two compound tones A and B are said to be *approximately consonant* when their dissonance thus reaches a minimum value. The following example will make this clear. Let the compound tone B be a very little higher in pitch than A. Then the fundamentals and every pair of overtones produce beats, and the dissonance is great. As the pitch of B is raised, this dissonance falls off as each pair of jarring overtones becomes more widely separated in pitch. This falling off in dissonance continues until the fifth overtone of B comes into the neighborhood of the sixth overtone of A; the beats of this pair of overtones then cause a distinct rise in the dissonance which is followed by a rapid fall as the two overtones come into exact unison. As the tone B continues to rise in pitch there is a rapid rise in the dissonance as the two overtones move apart in pitch, and so on.

The ordinates of the curve, Fig. 244, show the values of the dissonance of two violin tones A and B in so far as the overtones 2, 3, 4, 5, 6, and 8 are concerned. The tone A is kept at a constant pitch (see middle of figure), and the tone B starting with unison with A is at first slowly raised in pitch until its vibration frequency is twice as great as the vibration frequency of A; and then beginning again with unison with A, the tone B is slowly lowered in pitch until its vibration frequency is half as great as the vibration frequency of A. The fractions below the line BAB show the values of the frequency ratio B/A for which the various minima of dissonance occur. The numerical evaluation of dissonance is an approximation.*

Combination tones have some action in the production

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* See Helmholtz's *Sensations of Tone* (Ellis' translation), pages 272–299.

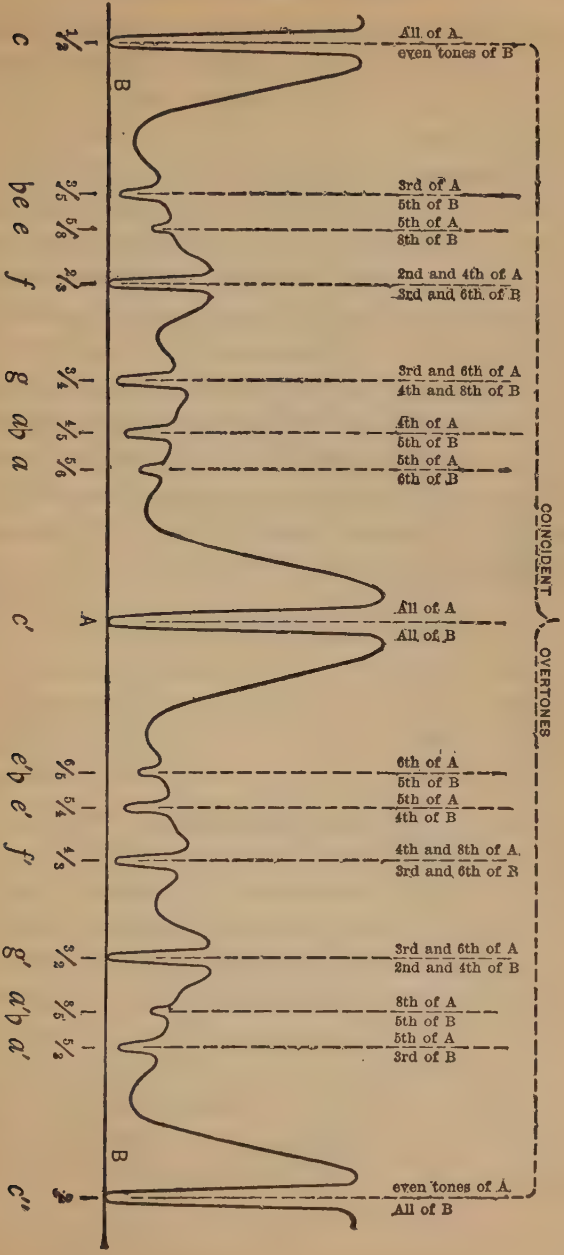

c
½
bc e
3/5 5/8
f
⅔
g
¾
ab a
4/5 5/6
c'
e'b e'
6/5 5/4
f'
4/3
g'
3/2
a'b a'
8/5 5/3
c''
2
B
A
B
All of A
even tones of B
3rd of A
5th of B
5th of A
8th of B
2nd and 4th of A
3rd and 6th of B
3rd and 6th of A
4th and 8th of B
4th of A
5th of B
5th of A
6th of B
All of A
All of B
6th of A
5th of B
5th of A
4th of B
4th and 8th of A
3rd and 6th of B
3rd and 6th of A
2nd and 4th of B
8th of A
5th of B
5th of A
3rd of B
even tones of A
All of B
COINCIDENT
OVERTONES

Fig. 244.

of dissonance when two tones are sounded together. The dissonance due to combination tones and dissonance due to overtones have minimum values for the same frequency ratios.

155. Consonant pitch-intervals. Two compound tones are said to be completely consonant when they are in unison, and they are said to be approximately consonant when their dissonance is at a minimum due to the coincidence of two or more overtones. When two compound tones are approximately consonant, the pitch interval between them is called a *consonant interval.* Thus, Fig. 244 shows the various consonant intervals, and the following table exhibits the various consonant intervals in the order of the completeness of their consonance. The names of the consonant intervals as given in this table are explained later.

TABLE OF CONSONANT INTERVALS

1 : 1 Unison.
1 : 2 Octave.
2 : 3 Fifth.
3 : 4 Fourth.
3 : 5 Major sixth.
4 : 5 Major third.
5 : 6 Minor third.
5 : 8 Minor sixth.

The bounding of consonant intervals. Those overtones and combination tones which determine a consonant interval by their coincidences, and which produce the greater part of the dissonance when the interval is slightly out of tune, are said to *bound* the interval. For example, the fourth and eighth overtones of *A* and the third and sixth overtones of *B* bound the interval *c′f′* as shown in Fig. 244, and the larger part of the dissonance of two tones which have nearly but not exactly the frequency ratio 4/3, is due to the jarring action of these particular overtones.

The great increase of dissonance due to a slight error of tuning of a consonant interval is the basis of our remarkably acute sense of the accuracy of tuning of these intervals. This acute sense of pitch in connection with consonant intervals has a great deal to do with the effectiveness of consonant intervals in music; for there can be no refinement of musical expression without an acute sense to seize upon it, and it is the ultimate dependence of this acute sense upon the presence of prominent overtones which explains the peculiar musical value of such tones as those of the violin and of the human voice.

156. Variation of the character of consonance with the quality of the tones. A consonant interval is the more striking in character in proportion as it is more sharply bounded (see Art. 155). Thus, the fifth (2 : 3) is a very sharply bounded consonant interval (see Fig. 244), whereas the minor sixth (5 : 8) is not sharply bounded, and the fifth is therefore much more striking in character than a minor sixth. The sharpness with which the various consonant intervals are bounded depends upon the relative loudness of the various overtones, and therefore the character of a given consonant interval varies with the quality of the tones used.

This is exemplified in a striking way by clarinet tones, which have only odd harmonics. The consonance of two such tones is most striking when it depends upon coincidence of odd harmonics of both tones, in fact, a consonant interval which depends upon or would depend upon the coincidence of even harmonics is bounded only by combination tones in the case of two clarinets. Thus, with two clarinet tones the major sixth (3 : 5) is more striking in character than the fourth (3 : 4), and, perhaps, even as sharply defined as the fifth (2 : 3); while the minor third (5 : 6) and the major third (4 : 5) are very poorly defined. A major third (4 : 5) formed by a violin tone and a clarinet tone is very much more striking when the clarinet tone is the lower than when the violin tone is the lower, because in the first case, the fifth overtone of the clarinet tone coincides with a fourth overtone of the violin tone, and in the latter case there is no fourth overtone of the clarinet tone to coincide with the fifth overtone of the violin tone. A minor third (5 : 6) is much more striking when the violin tone is the lower. The violin tone has both even and odd harmonics.

In the case of pure tones such as tones of tuning forks and broad organ pipes, combination tones, only, serve to bound the consonant intervals. With such tones the octave (1 : 2) is very sharply defined, the fifth (2 : 3) less sharply, and the remaining consonant intervals are scarcely bounded at all. Helmholtz, indeed, has found that the sound of two tuning forks is smooth or consonant whatever the pitch interval, provided the tones are not loud enough to bring out the combination tones strongly.

157. The major and minor triads. Two or more simultaneous tones which form a more or less consonant combination constitute what is called a *chord*. A chord which consists of three tones is called a *triad*. Any tone of a chord may be accompanied by its octave, or may be replaced by its octave, without greatly altering the character of the chord. This is evident when we consider that no new overtones are introduced into the sound by the octave.

The major triad and its modifications. The three tones of which the vibration frequencies are as **4 : 5 : 6** constitute what is called the major triad. By replacing the first tone (4) by its octave (8), we obtain a modification of this triad; and by replacing the first and second tones (4) and (5) by their octaves (8) and (10), we obtain another modification. The three forms of the major triad are, therefore,

4	5	6		
	5	6	8	
		6	8	10

The minor triad and its modifications. The three tones of which the vibration frequencies are as **10 : 12 : 15** constitute what is called the minor triad. The interval between the first two tones is a minor third (5 : 6), between the last two tones is a major third (4 : 5), and between the first and last the interval is a fifth (4 : 6); so that the minor triad contains the same consonant intervals as the major triad (4 : 5 : 6). The modifications of the minor triad are

10	12	15		
	12	15	20	
		15	20	24

The primary forms of the major and minor triads, namely, (4 : 5 : 6) and (10 : 12 : 15), are those in which the three tones are separated by the smallest pitch intervals.

Difference in character of major and minor triads. The major and minor triads contain the same consonant intervals, and the coincident overtones are identical in the two cases. The combination tones, however, are very different. The following schedule shows the combination tones of the first and second orders.

The major triad.

Primary tones,				4	5	6
First difference tones,	1	2				
Second difference tones		2	3	4	5	

The minor triad.

Primary tones,							10	12		15
First difference tones,	2	3	5							
Second difference tones,				7	8	9	10	12	13	

This schedule shows that the difference tones of the major triad are exact duplications, or duplications in the lower octaves, of the primary tones. That is, no foreign tones are introduced into the major triad by the combination tones. On the other hand, some of the difference tones of the second order, namely, 7, 8, 9 and 13, which occur in the minor triad, are dissonant, and give to this triad a character very different from that of the major triad.

158. Musical scales. The successive tones in a melody and the simultaneous tones in harmony are chosen with reference chiefly to their consonance. A collection of the notes which are available for melody or harmony is called a *musical scale.*

The major scale. Consider a given tone *c′*. The tones which can be used with *c′* with more or less consonance are those designated by *e′*♭, *e′*, *f′*, *g′*, *a′*♭, and *a′* in Fig. 244. Ignoring the tones *e′*♭ and *a′*♭, which have low degrees of consonance with *c′*, we have the following series of musical tones, each of which has some degree of consonance with *c′* :

Tones,	*c′*	*e′*	*f′*	*g*	*a′*	*c″*	I.
Vibration frequencies,	1	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	2	

The tone *c′*, with reference to which this series of tones is selected, is called the *tonic* of the series. The tone *g′*, having, next to the octave, the most complete consonance with *c′*, is called the *dominant*; and the tone *f′*, which is next in order of consonance, is called the *subdominant* of the series.

For purposes of harmony, it is desirable to be able to build major triads (4 : 5 : 6) upon the *tonic*, upon the *dominant*, and upon the *subdominant* of a series of musical tones. Two of these major triads may be built up with the tones in series I, namely, c', e', g' (4 : 5 : 6) and f', a', c'' (4 : 5 : 6). To build a major triad upon g', two additional tones, say b' and d'', are required such that $g' : b' : d'' = 4 : 5 : 6$. Therefore the vibration frequencies of b' and d'' are $\frac{15}{8}$ and $\frac{9}{4}$ respectively. Taking a tone d' an octave below d'', we have the series:

Tones,	c'	d'	e'	f'	g'	a'	b'	c''	II.
Vibration frequencies,	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2	

This is the ordinary musical scale, called the *major scale.*

The minor scale. Choosing, with the help of Fig. 244, the tones *below* c', which are most nearly consonant with c', we have the series:

Tones,	c	$e\flat$		g	$a\flat$	c'
Vibration frequencies,	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	1

or, in order that this series may be more easily compared with I, we may choose all of these tones an octave higher, whence we obtain the following series of musical tones:

Tones,	c'	$e'\flat$	f'	g'	$a'\flat$	c''	III.
Vibration frequencies,	1	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	2	

This series III is more melodious when sounded in the order of descending pitch than when sounded in the reverse order, for the reason that the tones of the series are more nearly consonant with c'' than with c', and whichever of these tones is sounded first is made correspondingly prominent. The series I (and also II) is more melodious when sounded in the order of ascending pitch.

The series III includes the two minor triads (10, 12, 15) c', $e'\flat$, g', and f', $a'\flat$, c''. To build a minor triad upon g', two additional tones, say $b'\flat$ and d'', are required, such that $g' : b'\flat : d'' = 10 : 12 : 15$, so that the vibration frequency of $b'\flat$ is $\frac{9}{5}$ and the vibration frequency of d'' is $\frac{9}{4}$. Taking a tone d' an octave below d'', we have the series :

Tones,	c'	d'	$e'\flat$	f'	g'	$a'\flat$	$b'\flat$	c''	IV.
Vibration frequencies,	1	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{9}{5}$	2	

This series of tones is called the *descending minor* scale. For purposes of melody this scale is changed to the following for ascending movements :

Tones,	c'	d'	$e'\flat$	f'	g'	a'	b'	c''	V.
Vibration frequencies,	1	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2	

This is called the *ascending minor* scale.*

Discussion of major and minor scales. The scales II and IV are better suited to the requirements of harmony than is scale V. The scale II is suited to harmony in which major triads predominate. It is for this reason called the major scale. The scale IV is suited to harmony in which minor triads predominate. It is for this reason called the minor scale.

The following schedules exhibit all of the major and minor triads which can be formed of the tones of major and minor scales.

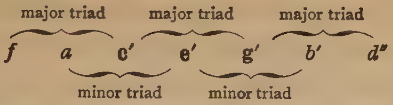
---

* Scale IV is called the descending *melodic* minor scale, scale V is called the ascending *melodic* minor scale ; a third minor scale which is more largely used than either of these in modern music is the *harmonic* minor scale, which is as follows :

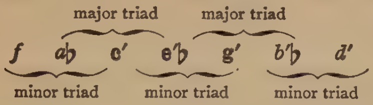
c' d' $e'\flat$ f' g' $a'\flat$ b' c''

The use of this scale results in part from employment in music of consonant effects depending upon the seventh harmonic (that is, the harmonic overtone which vibrates seven times as fast as the fundamental).

The major scale.



The minor scale.



The tonic triad is shown in each case by the bold-faced type. In the major scale the tonic triad, the dominant triad, and the subdominant triad are major triads. In the minor scale these triads are minor triads.

The naming of consonant intervals. The pitch intervals between the tonic and the third and sixth tones of the major scale are called the *major third* and *major sixth* respectively. The pitch intervals between the tonic and the third and sixth tones of the minor scale (IV) are called the *minor third* and *minor sixth* respectively. The intervals between the tonic and the fourth and fifth tones of either scale are called the *fourth* and *fifth* respectively.

Note. The tones which are consonant with the tonic *c′* are called *related tones of the first order.* The tones *d′* and *b′* of the major scale and *d′* and *b′*♭ of the minor scale which are consonant with *g′* are called *related tones of the second order* (that is, second order as related to the tonic *c′*).

159. Forms of musical expression. Expression of any kind depends upon the use of forms which may be readily seized upon and sharply distinguished by the senses (see Art. 155). The elementary forms of musical expressions are *tempo*, *rhythm*, *melody*, *harmony*, and *modulation*. The rapidity of succession of the tones in music is called the *tempo*; the manner in which successive tones are set off in accented groups is called *rhythm*; a sequence of tones is called a *melody*; and certain combinations of simultaneous tones constitute *harmony*.

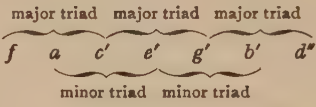
160. Modulation. Two chords are said to be related when they have one or more tones in common. Thus the two major triads

$$400 : 500 : 600 \quad \text{and} \quad 600 : 750 : 900$$

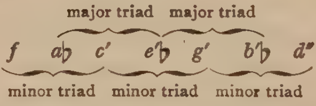
are related because they have the common tone 600. A sequence of chords, each related to the one preceding it, is called a *modulation*.

Examples. The following schedules, in which the tones are set off in major and minor triads, show the possible modulations in the major and minor scales.

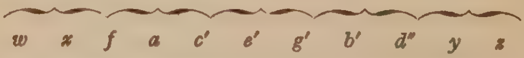
Major scale.



Minor scale.



More extended modulations than those exhibited in this schedule require the use of tones related, in the third order,* to the tonic c'. Let us consider, for example, an extension in both directions of the modulation of the major scale. We have:



The brackets represent major triads. The tones y and z are not related to c', but they are related to g' exactly as *b′* and *d″* respectively are related to *c′*. Thus the extension of this modulation upwards leads to a set of tones having a new tonic, namely, *g′*. In like manner the tones *w* and *x* are related to *f* exactly as *f* and *a* respectively are related to *c′*, so that an extension of this modulation downwards leads to a set of tones having a new tonic, namely, *f*. Such an extended modulation in one direction or the other is called a *change of key*.

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* See note at end of Art. 158.

161. The tempered scale. A great number of distinct tones is required for extended modulation, and it would be impracticable for a player to use a piano or organ having a separate key (and string or organ pipe) for each tone. This difficulty is overcome, at the expense of accuracy of tuning of the various consonant intervals, by the use of what is called the *tempered scale.* This consists of twelve tones (thirteen, counting both end tones) in each octave, the pitch intervals between successive tones being equal. The octave is thus divided into twelve equal pitch intervals. The logarithm of the frequency ratio of each of these intervals is therefore one-twelfth of the logarithm of 2, which is the frequency ratio of the octave (compare Art. 153). The accompanying table shows the logarithms of the frequency ratios of each tone of the major scale to the tonic, and also the logarithms of the frequency ratios of each tone of the tempered scale to the tonic.

This table shows that the first, third, fifth, sixth, eighth, tenth, twelfth, and thirteenth tones of the tempered scale are very nearly in unison with the successive tones of the untempered major scale. These tones may, in fact, be used for the tones of a major scale; and since the intervals of the tempered scale are all equal, it is clear that *any tone of the tempered scale may be chosen as* *a tonic, and that the third, fifth, sixth, eighth, tenth, twelfth, and thirteenth tones, counting from the chosen one, constitute a major scale.* All such major scales are equally well in tune. Unlimited modulations may be carried out on this scale inasmuch as any tone reached in a modulation has a group of tones related to it as a tonic. A minor scale may also be made up from the tempered scale. The tempered scale is now universally used.

COMPARISON OF TONES OF TEMPERED SCALE AND MAJOR SCALE.

Number of Tone in Tempered Scale.	1	2	3	4	5	6	7	8	9	10	11	12	13
Log. of ratio of tone of tempered scale to tonic.	0	0·0251	0·0502	0·0753	0·1003	0·1254	0·1505	0·1756	0·2007	0·2258	0·2509	0·2759	0·3010
Nearest tone in major scale.	*c*		*d*		*e*	*f*		*g*		*a*		*b*	*c′*
Log. of ratio of tone of major scale to tonic.	0		0·0511		0·0979	0·1250		0·1761		0·2219		0·2730	0·3010
Difference of logs.	0		− 0·0009		+0·0024	+0·0004		− 0·0005		+ 0·0039		+ 0·0029	0

CHAPTER XVII

MISCELLANEOUS PHENOMENA DEPENDING UPON THE REFLECTION, REFRACTION AND DIFFRACTION OF SOUND. ARCHITECTURAL ACOUSTICS

162. Echo. This familiar phenomenon is produced by the reflection of sound. The echo from the side of a large building is very clear and distinct, and the smooth face of a cliff or a well-defined forest front may produce an echo sufficiently distinct to repeat words. An echo grows less distinct the more irregular the reflecting surface, and it becomes a confused roar when the reflecting surface is very irregular. With multiple reflections, as in the case of the two walls of a cañon, a sharp loud sound, such as the report of a gun, becomes a prolonged rumble like thunder.

Reflection often produces the effect of apparent change of direction of a sound, when the direct waves from the source are masked or diverted so that the hearer perceives only the reflected wave-trains. This effect is sometimes strikingly produced at a street corner where the sound from an approaching trolley car is reflected from the wall of an opposite building.

163. Influence of the refraction of sound upon hearing at a distance.* Phenomena due to regular refraction, as sound waves pass from one medium into another in which the velocity is different, do not occur to ordinary observation. The following phenomena, however, which are due essentially to refraction, are of common occurrence.

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* A most interesting discussion of the transmission of sound through the atmosphere may be found on pages 257–323 of Tyndall's book, *On Sound.*

The velocity of the wind is usually less near the ground than higher up, and the upper portion of a sound wave *W*, Fig. 245, proceeding against the wind, is retarded. The direction of progression of the wave is thus turned upwards and the sound tends to leave the region near the ground. When the wave travels with the wind, the tendency is to concentrate the sound near the ground.

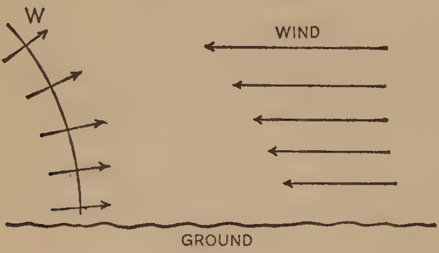

Fig. 245.

This is the explanation of the familiar fact that it is much more difficult to make one's self heard against than with the wind.

Sound travels faster in hot than in cold air. When the air near the ground is warmer than it is higher up, the upper portion of a sound wave is retarded and the sound tends to leave the ground. When the air near the ground is relatively cool, the tendency is for the sound to be concentrated near the ground. The greater distinctness of distant sounds by night than by day is due largely to this cause.

Influence of diffraction upon the sense of direction of a *sound.*[*] Our sense of the location (direction) of a source of sound seems to depend in part upon diffraction. The approaching waves reach one ear without much obstruction while the other ear is more or less shaded by the head. This shading action is greater the shorter the wave-length, so that a compound tone produces different sensations in the two ears. Differences of sensation brought about in this manner are significant of direction and have come to be perceived as such.

164. Acoustics of the auditorium.[†] The simplest auditorium is a level stretch of smooth, hard ground with a single hearer. In this case the sound of the speaker's voice spreads as a hemispherical wave and it decreases in loudness approximately as the inverse square of the distance. If the ground is covered by a large audience, the lower edges of the hemispherical sound waves are to a great extent absorbed, and thus greatly reduced in intensity; and the sound reaches the distant auditors largely by bending down from above, that is, by diffraction.

The first and most obvious improvement is to raise the speaker above the level of the audience, or to arrange the audience upon rising tiers of seats so that each auditor may receive sound by direct radiation, as it were, from the speaker. A second improvement is to place a reflecting wall behind the speaker so as to reflect more of the sound of the speaker's voice towards the audience. A third improvement is to arrange a roof so that the sound which radiates upwards may be reflected towards the audience and utilized. The closed auditorium thus arrived at is approximately perfect in so far as it directs practically the whole of the speaker's voice towards the audience, thus giving the maximum of attainable loudness for a given effort by the speaker The closed auditorium, however, gives rise to undesirable effects, confusion and distortion, which may be to some extent eliminated by proper design.

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[*] See Lord Rayleigh, *Philosophical Magazine*, Vol. 13, pages 214–232, February, 1907.

[†] The material in this and the succeeding articles is adapted from Professor W. C. Sabine's papers on Architectural Acoustics, *American Architect*, 1900.

Confusion. The closed auditorium gives rise to an overlapping of the successive sounds in speech or music. This is due to the prolongation of each sound by repeated reflections from the walls and ceiling of the room. If the room is very large these successive reflections of a given sound may be separately audible as a *multiple echo.* If the room is small, or if a large room has its reflecting surfaces broken up by pillars and alcoves, the successive reflections blend into a more or less continuous roar. This effect is called *reverberation.* In some auditoriums this reverberation lasts as long as five or six seconds after a loud sound, and when we consider that there are from fifteen to twenty separate articulate sounds produced by a speaker per second, counting both vowel and consonant sounds, we can understand what a serious matter excessive reverberation may be. In good theaters for the hearing of speech, the reverberation has a duration of from one to two seconds after a loud musical note, and in good music-halls the reverberation has a duration of from two to three seconds.

Distortion. Consider the various simple tones which enter into the composition of a compound tone such as a vowel sound. The wave-trains corresponding to these simple tones are reflected from the various walls of a room, and there are certain points in the room where one or more of the tones are strengthened while others may be weakened by interference. Thus, the relative loudness of the various simple tone components of the complex sound is changed and the character of the sound is altered. Furthermore, the air in a room has certain proper frequencies of vibration, the same as the air in an organ pipe; that is, the air in a room has certain proper tones, and any tone which is in unison with a proper tone of a room is more or less strengthened by resonance. Distortion by interference is most prominent in a room with flat reflecting walls, and distortion by resonance is most prominent in small rooms or in a room having alcoves. Distortion is seldom, if ever, a serious matter.

165. The elements of auditorium design. In the design of an auditorium provision must be made first of all for directing the sound of the speaker's voice, or the sound of an orchestra, towards the audience. Each auditor should be seated so as to receive sound by direct radiation from the source, and the reflecting surfaces, such as the ceiling and the wall back of the speaker, should be located so that the reflected sound may not reach the auditor more than, say, a twentieth of a second later than the sound which is radiated directly to the auditor; that is, the path of the reflected sound should not be more than about seventeen meters longer than the path of the directly radiated sound, otherwise the successive articulate sounds of a speaker will overlap perceptibly. Furthermore, the reflecting surfaces, to be effective, must be large in comparison with the wave-lengths of the prominently useful tones in speech and in music.

The walls of a large auditorium from which reflected sounds reach the auditors more than a fifteenth or a twentieth of a second after the directly radiated sound, should be broken up by pillars and alcoves so as to avoid sharp echo and the consequent confusing repetition of the articulate elements of speech.

Reverberation. For brevity, let us call those reflecting surfaces which send sound to the auditors with less delay than a twentieth of a second, *primary* reflecting surfaces; and those surfaces which reflect sound to the auditors with more delay than a twentieth of a second, *secondary* reflecting surfaces. All primary reflecting surfaces should be hard and smooth, so as to reflect as much as possible of the incident sound, and all secondary reflecting surfaces should be soft, covered with thick felt for example, so as to reflect as little as possible of the incident sound. The duration of reverberation could, of course, be reduced to the greatest possible extent by making all reflecting surfaces soft, but a serious sacrifice of loudness would be involved. The control of reverberation is therefore dependent upon the proper treatment of the secondary reflecting surfaces.

166. The equation of decaying sound in a room. The experimental determination of the absorbing power of various wall surfaces, and the use of such experimental data in the calculation of the duration of reverberation as a guide in the design of an auditorium, depends upon the equation which expresses the intensity i (sound energy per unit volume of the room) of a decaying sound at an instant t seconds after the source of the sound has ceased. This equation is

$$i = I\epsilon^{\frac{-.86a}{V}\cdot t} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

in which ϵ is the Naperian base and V is the volume of the room in cubic meters. The significance of the quantity a is as follows: The rate at which sound energy is lost in the room is proportional to the intensity of the sound at each instant: the rate at which sound energy would be lost through an open window is also proportional to the intensity of the sound at each instant. Therefore, the effect of walls and objects in a room in causing a loss of sound is *equivalent to a certain definite area, a square meters, of open window.* The walls and object in a room cause a loss of sound energy in three more or less distinct ways: (1) As the air particles near a wall oscillate to and fro parallel to the wall, sound energy is converted into heat by the friction. (2) When the air particles move to and fro perpendicularly to a porous wall, friction occurs in the pores of the wall and energy loss results. (3) A wall partakes more or less of the motion of the contiguous air, and if the material of the wall is imperfectly elastic energy losses occur.

Derivation of equation (i). It is required to find the number of units of sound energy which pass out of a room per second through a window of area a. Let V be the volume of the room and i the sound energy per unit volume at time t seconds after the cessation of the sound source. It is assumed that the sound energy is uniformly distributed throughout the room and that the sound in the neighborhood of a point is traveling, on the average, equally in every direction.

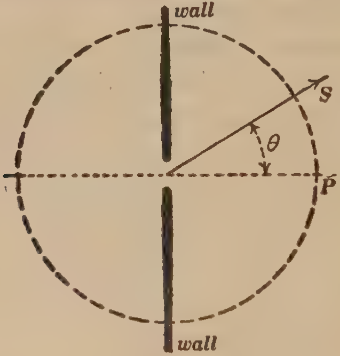


Fig. 246.

The problem under consideration is simplified if we consider the flow of energy in one direction through a hole in a wall built across the room so as to divide the room into halves. Describe a sphere of radius r with its center at the hole. The area of the sphere is $4\pi r^2$. We may imagine the sound energy per unit volume i, to be divided into $4\pi r^2$ equal parts, each of which parts is traveling at the normal velocity of sound (344 meters per second at ordinary room temperature) towards a separate unit of area of the sphere. Draw a diameter to the sphere perpendicularly to the wall. Consider the zone of the spherical surface which lies between θ and $\theta + d\theta$, where θ is the angle shown in Fig. 246. The area of this zone is $2\pi r^2 \sin\theta \cdot d\theta$. Therefore the fractional part, $2\pi r^2 \sin\theta \cdot d\theta / 4\pi r^2$, of the total sound energy near the hole is streaming towards the various elements of this zone and the component, normal to the wall, of the velocity of these streams of energy is $344 \cos\theta$. Therefore

$$\frac{2\pi r^2 \sin\theta \cdot d\theta}{4\pi r^2} \times i \times 344 \cos\theta \times a$$

is the rate at which energy is streaming through the hole towards the various elements of the zone. The total rate at which energy is streaming through the hole from left to right is therefore

$$E = 172ai \int_0^{\pi/2} \sin\theta \cos\theta \cdot d\theta$$

or

$$E = 86ai \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

This rate of flow of sound energy out of a room through a window of area a is equal to the rate, $d(Vi)/dt$, at which the total sound energy Vi in the room is decreasing, the room being assumed for the purpose of the immediate discussion to lose sound energy only through the window, therefore

$$\frac{d(Vi)}{dt} = -86ai \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

whence

$$i = I\epsilon^{\frac{-86a}{V} \cdot t}$$

in which ϵ is the Naperian base, and I is the value of i when t equals zero.

In an actual room the loss of sound energy due to absorption by the walls and objects in the room is sensibly *proportional at each instant to the intensity i of the sound* at that instant. Therefore equation (iii) expresses the rate of loss of total sound energy at each instant in an actual room, a being the open window area which is equivalent to the walls and objects in the room in so far as loss of sound energy is concerned, and equation (i) expresses the value of the decaying sound i at time t.

167. Absorbing power of various surfaces and objects.

The accompanying table gives the absorbing power of various surfaces and objects as determined by Sabine. These absorbing powers were determined by the methods explained in Art. 170.

To find the open-window-equivalent of a given surface multiply the area of the surface by p.

TABLE

Absorbing power of surfaces.

Character of Surface.	Equivalent Area (p) of Open Window for Each Square Meter of Surface.
Wood sheathing.	0·061
Plaster on wood lath.	0·034
Plaster on wire lath.	0·033
Window glass, single thickness.	0·027
Plaster on tile.	0·025
Brick set in cement.	0·025
Carpet rugs.	0·20 to 0·29
Cretonne cloth.	0·15
Shelia curtains.	0·23
Linoleum, loose on floor.	0·12
Hair felt, 2·5 cm. thick, 8 cm. from wall.	0·78
Audience, compact	0·96
Audience, compact, per person.	0·44
Isolated man (average).	0·48
Isolated woman (average).	0·54
Upholstered chairs, hair and leather.	0·30
Hair cushions, per seat.	0·21
Elastic-felt cushions, per seat.	0·20
Plain wood settees, per seat.	0 008

To find the open-window-equivalent of a number of similar objects, multiply the number of objects by the open-window-equivalent of one of the objects.

To find the open-window-equivalent of all the surfaces and objects in a room, add the open-window-equivalents of each. This gives the value of a in equations (i), (ii), and (iii).

168. Calculation of duration of reverberation. The degree of reverberation in a room is expressed by Sabine as the time t_1 required for a musical tone to decay from any initial loudness to a loudness one millionth as great. Let I be the initial loudness, then after t_1 seconds the actual loudness i will have decayed to the value

$$i = \frac{I}{1{,}000{,}000}$$

Substituting this value of i in equation (i) and solving for t_1 we have

$$t_1 = 0{\cdot}161 \frac{V}{a} \ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

Example. The New Music Hall in Boston has 1040 square meters of plaster on lath, which is equivalent to 34 square meters of open window, 1830 square meters of plaster on tile which is equivalent to 45 square meters of open window, 22 square meters of window glass which is equivalent to 0·6 square meter of open window, 625 square meters of wood which is equivalent to 38 square meters of open window, and the full audience including the orchestra is 2659 persons, which is equivalent to 1169 square meters of open window. Therefore, the value of a for this room with a full audience is 1286 square meters of open window. The volume of the room is 16,200 cubic meters. From these data equation (iv) gives 2·08 seconds as the value of t_1.

Remark 1. This example serves to show the predominating influence of the audience upon the duration of reverberation. If the seats are hard and smooth the value of a for the empty room would be about a fifth as great as in the above example, and the value of t_1 would be about ten seconds. If the seats have cushions, the value of a for the empty room would be about half as great as in the above example, and the value of t_1 would be about four seconds.

Remark 2. Consider two rooms A and B built on the same plan and of the same materials, room A being, say, three times as large in every dimension. Then the total absorbing power a of room A will be nine times as great as the absorbing power of room B, while the volume of A will be twenty-seven times as great as B. Therefore, the value of t_1 will be three times as great in room A as in room B. That is, reverberation is more and more pronounced the larger the room, and the importance of properly designing large rooms is correspondingly great.

169. Growth of sound in a room. When a source of sound, an organ pipe, for example, starts in a room, the intensity i of sound in the room increases until the sound is absorbed by walls and objects as fast as it is generated at the source. Let g be the rate at which the sound is generated and $-d(Vi)/dt$ the rate at which the sound is lost by absorption, then, when the sound has reached its full degree of loudness, we have

$$g = -\frac{d(Vi)}{dt}$$

Substituting this value of $d(Vi)/dt$ in equation (iii) and solving for i we have

$$i_u = \frac{g}{86a} \qquad \text{(v)}$$

in which i_u is the ultimate loudness or intensity reached by the sound in a room of which the total absorbing power is equivalent to a square meters of open window, and g is the rate at which sound energy is given off by the sounding body.

Remark. A number n of similar organ pipes similarly blown give off sound energy n times as fast as one pipe, and therefore the ultimate loudness of the sound produced in a given room is n times as great for the n pipes as it is for a single pipe.

170. Determination of absorbing power. *First method.* A standard source of sound, for example, an organ pipe supplied with air at constant pressure, is set up in a room having hard, smooth walls and provided with a number of adjustable windows. The material of which the absorbing power is to be determined is spread upon the walls or floor and the time that the sound remains audible after cessation of the source is observed. The material is then removed and the windows are opened until the sound remains audible for the same length of time as before. The total area of opening of the windows is then the measure of the absorbing power of the material.*

Second method. Determination of total absorbing power of a room. The time t'' for the sound of initial intensity I produced by one organ pipe to decay to minimum audible intensity i' is observed, and the time t''' for the sound of initial intensity nI produced by n similar organ pipes to decay to minimum audible intensity i' is observed. We then have from equation (i) the two equations:

$$i' = I\epsilon^{-\frac{86at''}{V}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a)$$

and

$$i' = nI\epsilon^{-\frac{86at'''}{V}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b)$$

Dividing equation (b) by equation (a) we have

$$1 = n\epsilon^{-\frac{86a}{V}(t'''-t'')}$$

or

$$0 = \log n - \frac{86a}{V}(t''' - t'') \log \epsilon$$

whence

$$a = \frac{V \log n}{86(t''' - t'') \log \epsilon} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{vi})$$

Third method. Determination of absorbing power of the different kinds of wall surfaces and objects. Suppose, for example, that there are four kinds of absorbing surface to be determined. Choose four different rooms, A, B, C, and D, in which these four surfaces occur in markedly different proportions.

Let A', A'', A''' and A^{iv}
B', B'', B''' and B^{iv}
C', C'', C''' and C^{iv}
D', D'', D''' and D^{iv}

---

* Some allowance should, of course, be made for the absorbing power of the bare wall or floor which is covered by the material.

be the respective measured areas of the four classes of reflecting surface in the respective rooms; let p', p'', p''' and p^{iv} be the absorbing power of each kind of surface, and let a', a'', a''' and a^{iv} be the total absorbing powers of the respective rooms. Then

$$\begin{aligned}
a' &= A'p' + A''p'' + A'''p''' + A^{iv}p^{iv}\\
a'' &= B'p' + B''p'' + B'''p''' + B^{iv}p^{iv}\\
a''' &= C'p' + C''p'' + C'''p''' + C^{iv}p^{iv}\\
a^{iv} &= D'p' + D''p'' + D'''p''' + D^{iv}p^{iv}
\end{aligned}$$

The four total absorbing powers a', a'', a''' and a^{iv} may be determined as explained above in the second method. Then these four equations permit the calculation of p', p'', p''' and p^{iv}.

APPENDIX A

CARDINAL POINTS AND CARDINAL PLANES OF LENS SYSTEMS

1. Centered system of thin lenses. A number of lenses used together constitute a *lens system*. When the centers of curvature of the various spherical surfaces of the lenses lie on one straight line, the system is called a *centered system*. The following articles treat of the action of a system of thin lenses on the assumption that the various imperfections (see Chapter VI) are negligible, and the discussion is limited to the consideration of pencils of rays which are but very slightly inclined to the axis of the system. The following figures, however, show rays making considerable angles with the axis for the sake of clearness.

Inasmuch as the individual thin lenses are assumed to be free from any imperfections, the following propositions are evidently true. The following discussion of cardinal points and cardinal planes is based upon these propositions.

(*a*) A narrow pencil of rays from any luminous point O in or near the axis of a lens system, is sensibly homocentric after passing through the system, and the emergent pencil is concentrated at or appears to have come from a point O'. Two such points are called *conjugate points*.

(*b*) A group of luminous points (an object) near the axis and in a plane perpendicular to the axis, has, as its image (with definite magnification, positive or negative) a similar group of points near the axis in another plane perpendicular to the axis. Two such planes are called *conjugate planes*.

Cor. 1. Any incident ray passing through a point O must, upon emergence, pass through O', the conjugate to O. An incident ray and the corresponding emergent ray are called *conjugate rays*.

Cor. 2. Two incident rays intersecting at O, intersect, upon emergence, at O'. A particular case is where one incident ray is coincident with the axis of the system. Such an incident ray emerges without change of direction, and therefore the points where any two conjugate rays intersect the axis constitute a pair of conjugate points.

Cor. 3. An incident ray which passes through the points O and p must, upon emergence, pass through O' and p', the conjugates to O and p.

Note 1. If an emergent ray be reversed, for example, by normal reflection from a mirror, it will retrace its path. If, therefore, the ray r' is conjugate to the ray r, then r is conjugate to r'; and if the point O' is conjugate to O, then O is conjugate to O'.

Note 2. Any specification which completely determines the position upon emergence of any given incident ray, is a complete specification of the lens system.

2. Specification of a lens system. *A lens system is completely specified, in so far as its optical properties are concerned, when the positions of two pairs of conjugate planes are given together with the magnification associated with each pair.* The magnification is the ratio of diameter of object to image, and it is considered to be negative when the image is inverted.

Proof. Let aa' and bb', Fig. 1, be two given pairs of conjugate planes. Let m be the magnification associated with aa' and let m' be the magnification associated with bb'. Figure 1 is constructed for $m = -1\cdot 5$ and $m' = +1\cdot 5$, that is to say, any point in plane a is m times as far from the axis as its conjugate in plane a', and any point in plane is m times as far from the axis as its conjugate in b'. Let r' be any given incident ray, cutting the planes a' and b' at the points p' and q' as shown. The emergent ray r must pass through the points p and q which are conjugate to p' and q', respectively. The emergent ray is thus determined, and the action of the lens system upon the incident ray r' is therefore completely established by the given data.

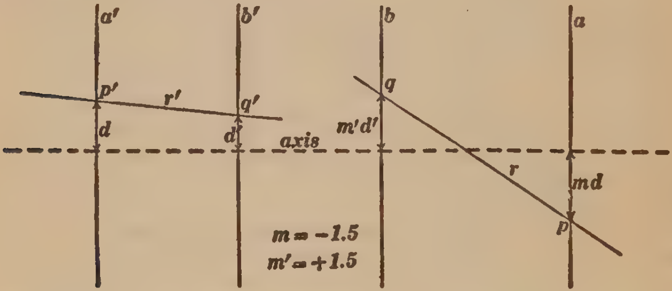


Fig. 1.

3. Focal points of a lens system.* Consider an incident ray r', Fig. 2, coming from the left parallel to the axis. Its conjugate ray r is as shown. If the distance d changes, the distances md, $m'd$ change in the same ratio, as shown by the dotted rays R and R', and a certain point F remains fixed (in Fig. 2 the point F is midway between the planes a and b because the magnifications m and m' are equal and opposite). Therefore, all rays from the left, parallel to the axis, pass through the point F, or seem to have come from F after emergence.

---

* What are usually called *principal foci* are here referred to.

This point F is called the *right focal point* of the system. Figure 3 shows the construction for the *left focal point*, F' (in Fig. 3 the point F' is midway between the planes a' and b' because the magnifications m and m' are equal and opposite).

4. Principal planes of a lens system. Those two conjugate planes for which the magnification is *plus one* ($+1$) are called the *principal planes* of the system. The following discussion shows that there is always such a pair of planes, and shows their location relative to the given conjugate planes aa' and bb'.

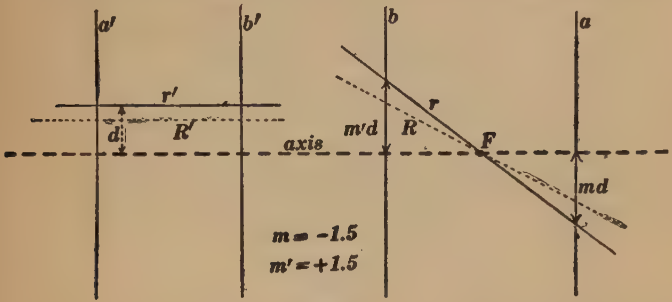


Fig. 2.

Let r, Fig. 4, be an incident ray from the right and let r' be its conjugate. Let R', colinear with r, be an incident ray from the left and let R be its conjugate. The rays r and R intersect at o, and the rays r' and R', which are conjugate to r and R, respectively, intersect at o'. Therefore o and o' are conjugate points, and P and P' are conjugate planes; and since o and o' are at the same distance from the axis, it follows that the magnification for the conjugate planes P and P' is unity, so that P and P' are the required principal planes of the system. The distance P to F is called the *right focal length* of the

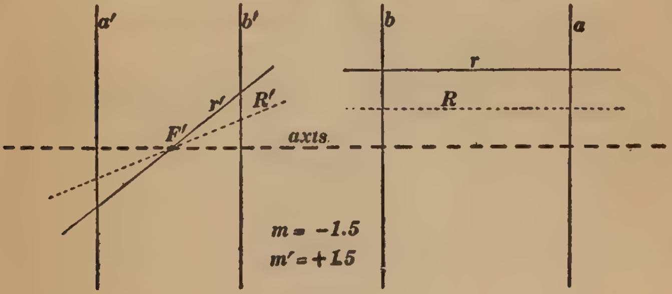


Fig. 3.

system, and the distance P' to F' is called the *left focal length* of the system. The system represented in Figs. 1, 2, 3, 4, 6, and 7 is a converging system because the right focal point F is to the right of the principal plane P, and the left focal point F' is to the left of the principal plane P', that is to say, both focal lengths are positive. The two focal lengths of the lens system shown in Fig. 4 are unequal. The two focal lengths of a lens system have a ratio equal to the ratio of the refractive indices of the media in which the respective focal points lie. In most lens systems used in practice, the two focal lengths are equal since both focal points are in air.

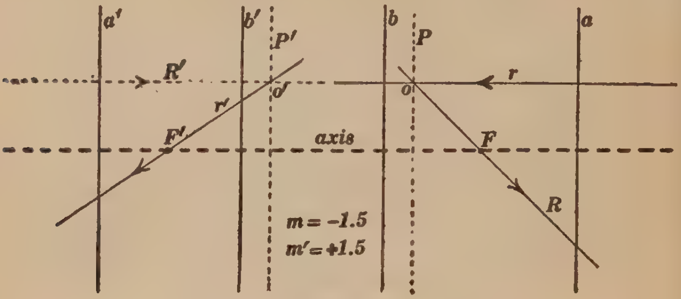


Fig. 4.

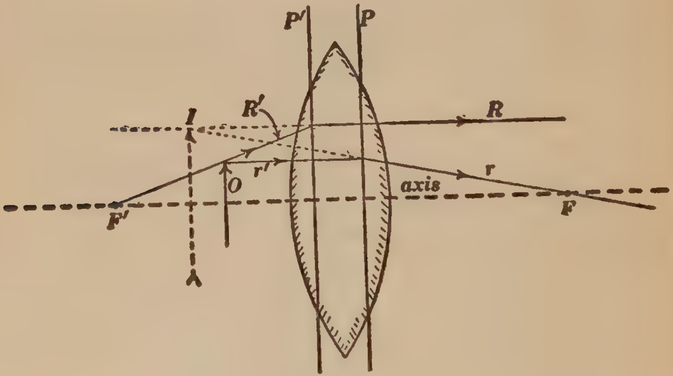


Fig. 5.

Examples. The plane LL in Fig. 127 of Chapter VII, is the right principal plane of the lens system shown in that figure, and the distance from LL to F is the right focal length of the system. The plane LL is determined, as in Fig. 4, by the intersection of the emergent ray R

with the line of the incident ray R' (extended if necessary). The point A in Fig. 128 shows the position of the right principal plane of the lens system for red light, and the point B shows the position of the right principal plane for violet light.

Figure 5 shows to scale the actual positions of the principal planes of a symmetrical bi-convex lens (not of infinitesimal thickness), the glass of which has a refractive index of 1·5. The figure also shows the geometrical construction for determining the position of the image of a given object as follows: Draw the ray r' parallel to the axis from O to the principal plane P, and thence draw the ray r through the focal point F. Draw the ray R' from F' through O to P', and thence draw the ray R parallel to the axis. The point of intersection of the emergent rays r and R is the conjugate of O and it determines the position of the image. It is instructive to compare this geometrical construction with that which is given in Art. 50 of Chapter V for a simple thin lens.

5. The inverse principal planes of a lens system. Those two conjugate planes for which the magnification is *minus one* (— 1) are called the *inverse principal planes* of the system. The following discussion shows that there is always such a pair of planes, and shows their location relative to the direct principal planes P and P' and the focal points F and F' of the system. Let the principal planes P and P' and the focal points F and F' of the system be given as shown in Fig. 6. These elements completely specify the system. Draw the ray R from the right parallel to the axis of the system, and draw its conjugate ray R' through the left focal point F', as shown. Choose the point q' on the ray R' at the distance d below the axis. The conjugate to q' is, of course, on the ray R, and therefore at the same distance d above the axis. Therefore, the plane Q' passing through q' is one of the inverse principal planes. To determine the other inverse principal plane, draw the incident ray r' from the left and its conjugate ray r passing through the right focal point of the system. The point q is conjugate to the point q' and therefore the plane Q is the other inverse principal plane.

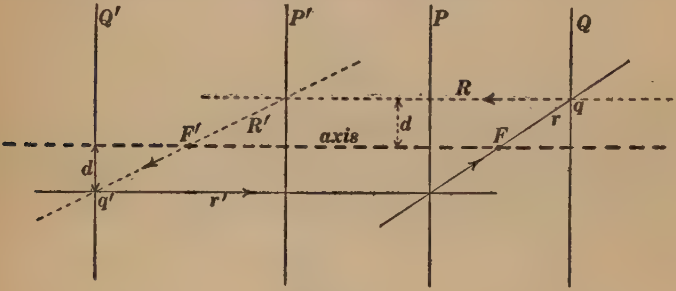


Fig. 6.

From Fig. 6 it is evident that the distance PQ is equal to $2f$ and that the distance $P'Q'$ is equal to $2f'$, where f and f' are the right and left focal lengths of the system, respectively.

6. Nodal points of a lens system. *There are two conjugate points in the axis such that any incident ray passing through one of these points is, upon emergence, parallel to its direction upon incidence.* These two conjugate points are called the *nodal points* of the system. To understand the geometrical significance of the nodal points, consider an object and its image as shown in Figs. 7 and 8. The various rays from the object which pass through the nodal point N' radiate, upon emergence, from the nodal point N, the direction of each emergent ray being parallel to the direction of that particular ray upon incidence, so that the object as seen from N' is of the same angular magnitude as the image as seen from N, or in other words, the diameter of object

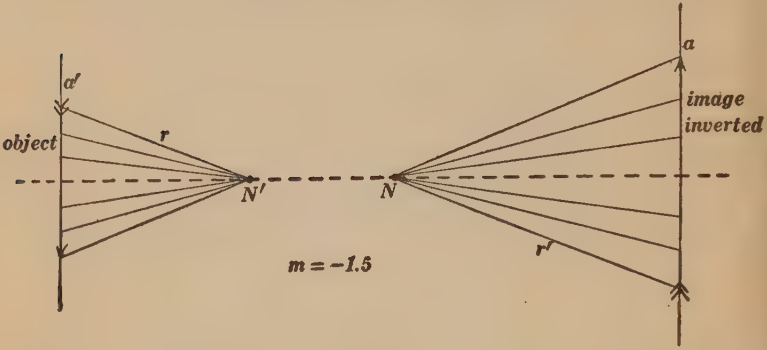


Fig. 7.

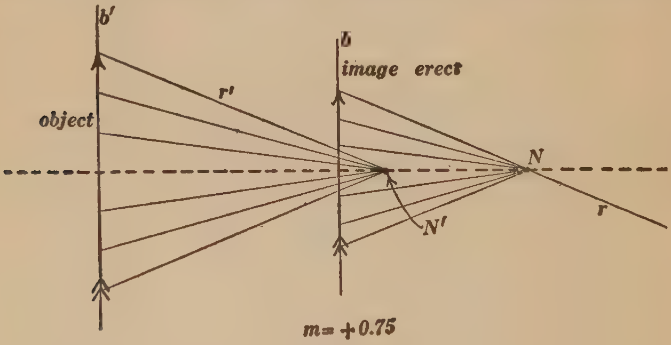


Fig. 8.

and the diameter of image are directly proportional to their distances from the respective nodal points.

Consider an object in one of the principal planes P' and its image in the principal plane P. The image is the same size as the object and erect, and therefore the nodal points must be at the same distance and in the same direction from the respective principal planes (see Fig. 9). Consider an object in the inverse principal plane Q and its image in the inverse principal plane Q'. Inasmuch as the image is inverted and of the same size as the object, the two nodal points N' and N must be at the same distance but in opposite directions from the respective inverse principal planes Q' and Q (see Fig. 9).

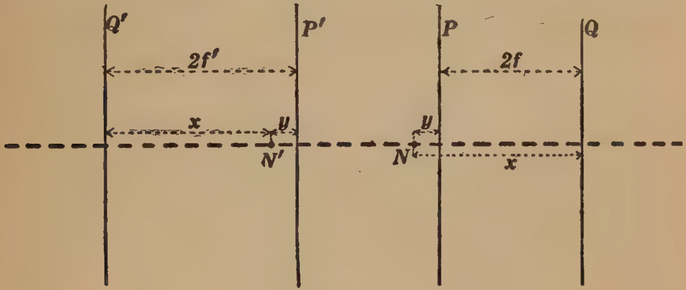


Fig. 9.

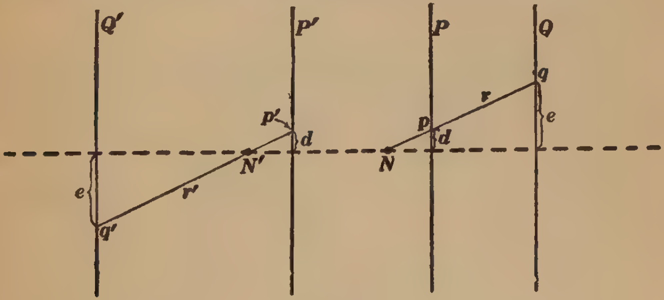


Fig. 10.

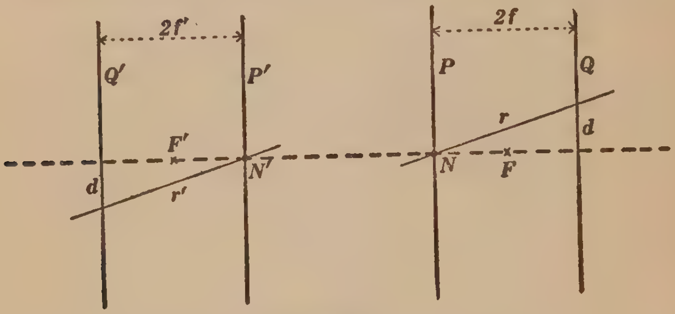
The values of x and y in Fig. 9 satisfy the equations:

$$x + y = 2f'$$
$$x - y = 2f$$

whence

$$x = f + f'$$
$$y = f' - f$$

It remains to show that the points N' and N are conjugate to each other. Draw the two parallel lines r' and r in Fig. 10. Since the distances $N'Q'$ and NQ are equal, and since the distances $N'P'$ and NP are equal it is evident that the two points p and p' are at the same distance above the axis, and that the two points q and q' are at equal distances on opposite sides of the axis. Therefore q and q' are conjugate points and p and p' are conjugate points, so that r and r' are conjugate rays, and therefore the points N and N', where r and r' cut the axis, are conjugate points.



when $f' = f$
Fig. 11.

When the right and left focal lengths of a lens system are equal as shown in Fig. 11, then the nodal points N' and N lie in the principal planes.

7. Focal length of a lens system (doublet) expressed in terms of focal lengths of individual lenses and their distance apart. The discussion of this matter is greatly facilitated by considering the curvature of a wave surface at different points as it passes through the system. The curvature of a spherical surface may be expressed as $1/r$, where r is the radius of the sphere. Thus, a sphere of 100 centimeters radius has a curvature of 0·01 per centimeter. A wave like WW, Fig. 12, may be thought of as having a *positive curvature* and a wave like WW, Fig. 13, may be thought of as having a *negative*

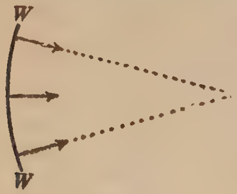


Fig. 12.

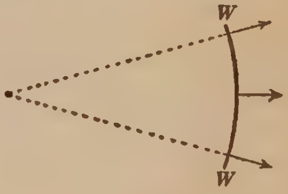


Fig. 13.

curvature. The curvature of WW in Fig. 12 is continually increasing in value because its radius is decreasing, and the curvature of WW in Fig. 13 is decreasing in value because its radius is increasing.

The effect of a simple converging lens is to *add* a definite amount $1/f$ to the curvature of the wave, where f is the focal length of the simple lens. In the case of a diverging lens, the focal length is considered negative, and the effect of such a lens is to subtract $1/f$ [or to add $(-1/f)$] from the curvature of a wave. These propositions may be established by a careful consideration of the substance of Art. 51 of Chapter V.

Examples. A wave reaches a simple converging lens from a point a distant u from the lens so that the curvature of the wave when it reaches the lens is $-1/u$ (see above discussion of positive and negative curvature). The lens adds an amount $1/f$ to the curvature of the wave, making it $1/f - 1/u$, which is, of course, equal to $1/v$, where v is the distance from the lens to the point b, which is the conjugate of a. Therefore we have $1/f - 1/u = 1/v$, or $1/f = 1/u + 1/v$, which is equation (i) of Art. 51 of Chapter V.

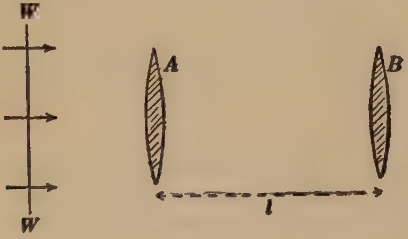


Fig. 14.

Consider two simple lenses A and B, Fig. 14, of which the respective focal lengths are f_1 and f_2, l being the distance of the lenses apart. Consider a plane wave WW (of zero curvature). After passing the first lens A, the curvature of this wave is $1/f_1$. When this spherical wave reaches the lens B, its radius of curvature has decreased to $f_1 - l$, and therefore its curvature is $1/(f_1 - l)$. When the wave passes through lens B its curvature is increased by the amount $1/f_2$ and the curvature then is $1/(f_1 - l) + 1/f_2$, and this is evidently equal to $1/D$, where D is the distance beyond B to the point F, where the wave WW is brought to a focus by A and B, as shown in Fig. 15. Therefore, solving $1/D = 1/(f_1 - l) + 1/f_2$ for D, we get $D = (f_1 - l)f_2/(f_1 + f_2 - l)$.

Figure 15 shows the right principal plane PP of the combination AB, and the distance f_c is the focal length of the combination. A careful inspection of Fig. 15 gives

$$D/f_c = (f_1 - l)/f_1$$

whence, using the above expression for D and solving for f_c, we get

$$1/f_c = 1/f_1 + 1/f_2 - l/f_1f_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

or, using p_c for $1/f_c$, p_1 *for* $1/f_1$, and p_2 for $1/f_2$, we get

$$p_c = p_1 + p_2 - p_1p_2l \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

The partial achromatization of the* Huygens and Ramsden eyepiece doublets. These doublets are shown in Figs. 135 and 136 of Chapter VII. In each case $l = \frac{1}{2}(f_1 + f_2)$, where l is the distance between the lenses of the doublet and f_1 and f_2 are the respective focal lengths of the individual lenses. *The focal length f_c of the combination is in both cases the same for all colors.* This may be shown as follows:

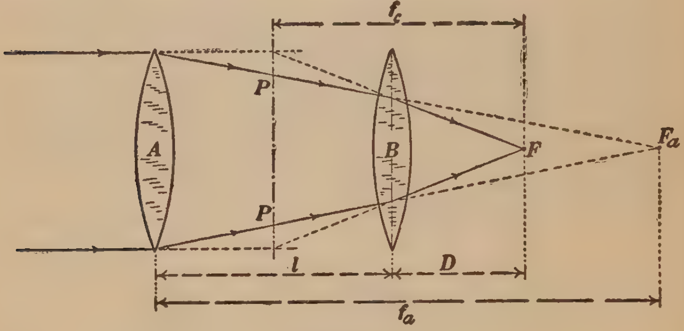


Fig. 15.

From equation (i) of Art. 52 of Chapter V we have

$$d\left(\frac{1}{f}\right) = \frac{1}{f}\left(\frac{d\mu}{\mu-1}\right) \text{ or } dp = p\left(\frac{d\mu}{\mu-1}\right) \quad \ldots\ldots\ldots\text{(iii)}$$

but $d\mu/(\mu - 1)$ is the same in value for both lenses of a Huygens or Ramsden doublet because both lenses of either doublet are made of the same kind of glass. Therefore, differentiating equation (ii) and using $dp_1 = p_1\left(\frac{d\mu}{\mu-1}\right)$ and $dp_2 = p_2\left(\frac{\mu}{\mu-1}\right)$, we get

$$dp_c = (p_1 + p_2 - 2p_1p_2l)\left(\frac{d\mu}{\mu-1}\right) \quad \ldots\ldots\ldots\ldots\text{(iv)}$$

but if $l = \frac{1}{2}(f_1 + f_2) = \frac{1}{2}(1/p_1 + 1/p_2)$, we have $p_1 + p_2 - 2p_1p_2l = 0$ and therefore equation (iv) shows that the change of p_c with μ (due to a change of color) is zero. That is to say, p_c (and, of course, f_c) is independent of color.

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* Meaning in this case achromatizing for focal length only. See Art. 77 of Chapter VII.

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